Jiří Gregor
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Discovering Mathematics

A Problem-Solving Approach to Mathematical Analysis with Mathematica® and Maple™

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'...mathematics has two faces; ... Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science. Both aspects are as old as the science of mathematics itself.'

G. Polya

'A problem? If you can solve it, it is an exercise; otherwise it's a research topic.'

R. Bellman

Prologue

The best way to learn things is by doing them. But why mathematics?

Mathematics was always one of the less popular topics in school, mathematics is not easy to study, mathematics teachers are demanding and answers to their questions are often encoutered with objections, problems they pose are difficult to solve, etc.

Is there a way to override negative sentiments towards mathematics?

Success is a good stimulus. We have to know how to solve problems successfully and then they may become fascinating and rewarding. Some people want to start with difficult problems and follow some suggested path to a solution, others may want to gradually develop their ability on less complicated problems.

An ability to solve problems will certainly be useful in everyday life.

An interconnected net of mathematical problems from various sources and with various levels of difficulty is presented together with supporting material (hints, plans of solution, definitions and theorems, answers and references) and any student, teacher, engineer or interested person may sharpen his skill in his own way.

Any new piece of knowledge can be built only on earlier acquired knowledge.

The way of applying Calculus in this collection requires some basic knowledge (a first year university course is sufficient). To show interesting problems which are usually not included in Calculus courses, the ordering of topics is rather non-standard. Three main parts are included: Concepts, Tools, Applications.

Do we need mathematics when 'all problems can be solved by computers'?

This common statement is known to be misleading. But computers with powerful software can be a valuable tool when we want to get rid of routine calculation, to verify conjectures on examples or visualize results. We have chosen MATHEMATICA and Maple as the software tool and some hints are given on where and how it can be used.

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Part I
Concepts

Introduction

Why Mathematics?

The solution of complex problems in engineering, in economics and even in everyday life is often very difficult and accepting results which have not been verified to sufficient extent may sometimes be rather risky. We cannot hope for a 'general solving method', nevertheless some kind of strategies or structured anticipations—'the art of problem solving'—can be described. Experiments, techniques of trial-and-error, methods based on simplification, analogy and abstraction are recognized as indispensable parts of this art. In science and technology such tools are often successfully replaced by mathematical models of the problem under investigation. Roughly speaking, a mathematical model is a collection of concepts with well-defined properties and interrelations such that the behavior of the collection is apt to imitate the behavior the object under investigation. Such a mathematical structure is a 'language' in which the problem has to be expressed. Such expression is the mathematical formulation if the problem.

The loading capacity of a crane or a bridge can be calculated by solving certain differential or algebraic equation, a system of linear equations and inequalities is used in solving optimization problems, a weather forecast can be obtained as a solution of a system of partial difference equations, the Rubic cube can be solved using some knowledge of group theory, some methods of coding can be described by applying number theory, and many more examples can be given.

Two Stages

Some standard problems have standard mathematical models. For instance, dynamical processes are usually described by ordinary differential equations in various settings. To solve geometrical problems, various coordinate systems and methods of analytic or algebraic geometry can be used. On the other hand, some problems lead us into rather unexpected parts of mathematics, or—as shown in the history of mathematics—they may lead to building new branches of mathematics. Analysis of games lead to probability theory, problems

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like the famous 'Seven bridges of Königsberg' leads to graph theory. We have to conclude that when trying to establish a mathematical model we cannot be restricted to a predefined area of mathematics. A mathematical model can be rather difficult to find, often because specialists speaking different 'languages' of their separate branches of science must understand each other, find a common language and formulate the ultimate goal.

In a mathematical modeling approach to problem solving, we need to discuss two distinct activities. First of all a suitable mathematical model and a mathematical formulation has to be found and secondly, we must be able to solve the emerging mathematical problem.

Solving some complicated scheme of trains moving at different speeds in opposite directions, with stops and restrictions, we assume that interrelations between speed, distance and time are known. The main problem is to establish a system of equations governing all the relevant quantities. Their solution might be rather simple, although some supplementary steps of verification, exclusion of some virtual solutions and other concluding steps might still cause difficulties. Assuming that we have a mathematical formulation of our problem, we have to make sure that its solution exists (in many cases this is far from obvious) or perhaps that it has many solutions. Then we can try to find its solution(s). Software packages like MATHEMATICA®[1], Maple®[2] or others can be of substantial help. Interpretation of the result and its verification is the final part of the solution.

Problem Solving

The only way to learn the art of problem solving is by doing it. Mathematical reasoning contains all the necessary constituents of general problem solving and this is the main reason for studying mathematics. A common difficulty is that mathematical textbooks often contain (ready-made) definitions and theorems without motivation or justification. But we have to know why the introduced results are the most adequate and useful concepts, and why other possible ways are wrong or misleading. Also exercises often, by their location or context, have predefined steps of solution. These features do not support the skills of problem solving and a more general approach is desirable. A prerequisite of such an approach is a basic knowledge of mathematics (e.g. calculus, geometry and algebra). On such a basis we may try to look behind the curtains of mathematics: to learn how and why various mathematical concepts are defined, to acquire basic skills in using mathematical tools and devices, and finally to apply such knowledge to solve problems which often have their origins outside mathematics.

[1]Mathematica® is a registered trademark of Wolfram Research, Inc., 100 Trade Center Drive, Champaign, IL 61820-7237, USA, http://www.wolfram.com

[2]Maple® is a trademark of Waterloo Maple Inc., 615 Kumpf Drive, Waterloo, Ontario, Canada N2V 1K8, http://www.maplesoft.com

How To Solve It

Probably one of the best description of the process of problem solving with an emphasis on mathematics has been given by G. Polya in his book 'How to solve it'. The author describes the following four basic stages of problem solving in detail:

1. Understanding the problem.
2. Devising a plan.
3. Carrying out the plan.
4. Looking back.

In all these stages the author recommends to answer some questions and carry out small tasks. In a shortened rewording some of them can be formulated as follows:

Do I understand what is wanted, what is known and what the conditions are?

Are the conditions sufficient, insufficient or redundant?

Do I know the solution of a similar, related or analogous simpler problem?

Can I use its result or its method? Can I design successive steps towards a solution? If not, can I formulate and solve a simpler related problem and use the result?

Can I carry out the designed steps towards a solution?

Can I check the result of each of the steps?

Did I use all the given data and did I satisfy all the conditions?

Are there other solutions?

Can the obtained result be used in solving other related problems?

To illustrate this approach let us try to solve the following task:

Find the point which is the nearest to n given ones.

Almost all the words in this formulation are ambiguous and we cannot even start thinking of how to solve it. We have to analyze its formulation. The most suspicious part seems to be the wording 'the nearest'. It might be connected to distances; one possible interpretation could be: the point for which the sum of distances from the n given ones is the smallest. But are all the given points mutually distinct? If not, should the distance from the unknown point to two coinciding ones be counted twice? Further, what is meant by 'point'? Are they points in the geometrical sense, or are they elements of some abstract space? How are they defined in the latter case? Last but not least: how is the distance between two points measured (or defined)?

To resolve these ambiguities we may want to reformulate the question. We recall that the concept of distance has been abstractly defined. A set such that any two of its elements, say A and B, have a defined distance $\rho(A, B)$ is commonly called metric space and elements of such a set can be called points. Therefore the above task can be reformulated as follows: Given n mutually distinct elements of a metric space, find an element x of this space such that the quantity $\sum_{i=1}^{n} \rho(x, x_i)$ (i.e. the sum of the distances from x to the given x_i's) is the smallest possible. In such an abstract formulation the problem seems to be unsolvable. Looking back at our starting point we may feel that originally the points are in fact points in common 'space', i.e. in three-dimensional space, in which the distance means the Euclidean distance. Hence, for the distance $\rho(A, B)$ we have

$$\rho^2(A, B) = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2$$

for all $A, B \in \mathbf{E}^3$, where x, y, z are the coordinates of A and B. By accepting the last specialization we exclude cases which might be important, e.g. our task is not solved for points on the earths surface etc. But even with this restriction you will probably find the solution of this problem challenging.

We could proceed by lowering the dimension of the space, i.e. solve it for points on a plane or on a straight line. In the solution process we may discover that the main difficulty is the square root involved. We can go back in the line of reasoning, and change the formulation to get rid of the square root, which means adopting the following formulation:

Given n mutually distinct points $A_i, i = 1, 2, \ldots, n$ in three-dimensional Euclidean space $\mathbf{E}^3$. Find, if it exists, a point A_0 such that

$$\sum_{i=1}^{n} \rho^2(A_0, A_i) \leq \sum_{i=1}^{n} \rho^2(A, A_i)$$

for all $A \in \mathbf{E}^3$.

The original task has now turned into a well-formulated problem and its solution is rather straightforward. Taking partial derivatives of the right-hand side, putting them equal to zero, we easily obtain the solution. As a bonus we obtain that the desired point is the center of gravity of the given points. Also we now have a much better insight into the difficulties, obstacles and ways of solution of the original problem.

Turning back in our reasoning we may ask whether minimization of the sum of distances and minimization of the sum of squared distances yields identical results. It does not (see Problems Q 29 and V 09 below) and this also illustrates how easily ambiguity can lead to misinterpretation of a problem.

It can be seen that finding the solution of a given problem often consists of (i) generating an interconnected net of simpler well-formulated problems, (ii) finding the solvable ones in this net and (iii) generalizing the solutions, the methods, the formulations of these problems and step-by-step attain a solution of the original problem.

As an example let us assume that we want to analyze the behavior of a church bell. This includes its motion caused by external forces, its sound, shape etc. To simplify the problem of its motion we could consider separately the bell and its tongue. The tongue can be simplified to a physical pendulum. Further abstraction of the shape, mode of suspension etc. leads to the concept of the mathematical pendulum. Its motion can be described using Newton's law by a nonlinear differential equation (see M 27 below). Here either its solution can be looked for by various methods of discretization or we might follow the path of linearization restricting ourselves to 'small' initial deviations. The last problem is actually quite simple. When the sound of the bell is important, we may start by analyzing the sound of a metal plate, and further restrict the problem to the sound of a string so that we finally find an equation, which actually resembles the one found when investigating the motion of the tongue. Again, its solution is simple. Turning back to the motion of the bell, we might need to find the center of mass of the bell, which leads back to the shape of the bell.

Although this example is extremely simplified, it illustrates the basic idea and the basic procedure of problem solving: we must be able to formulate as precisely as possible various 'parts' of the problem and make sure that we can solve at least the simplest one.

As a further example, consider a simplified problem of cartography. In a planar region (i.e. in a small area) we can fix a number of points and measure some of the distances

and angles defined by these points. In fact, we consider a set of triangles and the measured quantities may or may not be sufficient for calculation of some unknown distances. Commonly, the number of measurements is higher than necessary and the obtained data form a system of linear equations, which is overdetermined, i.e. no solution of the system exists. On the other hand, the data are distorted by errors of measurement. Mathematics gives a method, called the least square technique (or in another treatment pseudoinverse matrices), which enables us to find such values of the unknown quantities which 'satisfy' the overdetermined system of equations in the best possible way. Such 'pseudosolution' can also minimize the effect of the errors of measurements. The necessary step here was to redefine the mathematical concept of solution of a system of equations, i.e. to redefine the mathematical model adopted for the solution.

In the description above the reader will easily point out the places where computers and software packages can help.

Discovering Mathematics

The above-described scheme of mathematical problem solving led to the basic idea of this collection. Mathematical problems, mostly drawn from calculus, are presented in a form which enables, and often demands, to create an interconnected net of simpler problems. The reader is encouraged to find his own simplifications, reformulations and interconnections in the solved task.

In this collection the problems are interconnected by hyperlinks (indicated by vertical up or down arrows). In fact, up arrows can be considered as hints towards easier and related problems, down arrows point towards more interesting and more involved problems. These hyperlinks are intended as a help, but they also show how more difficult problems can be simplified. For each problem a rough estimate of its difficulty is indicated by one, two or three dots •. Definitions and basic related theorems are available via hyperlinks and hints for a solution are occasionally attached. In more difficult problems a hyperlink, marked ↓P xx, points towards a suggested plan of solution. Related problems in different chapters are also connected by hyperlinks. All the hyperlinks in the MATHEMATICA® or Maple® versions are printed in the ESM in blue and underlined.

To avoid lengthy calculations, but first of all to examine special cases quickly and conveniently, verify conjectures and/or obtain illustrative pictures, software packages can be used. MATHEMATICA® and Maple® are the most common ones. Problems where the use of such software is advisable or indispensable, are marked [M]. Commands of these packages are written in accordance with syntax rules and given in the ESM in red in the text and in the Appendix together with the final results. The software is supposed to help the user to concentrate on the basic mathematical content, suggest and check some partial results, give better insight to the content of the problem, avoid the burden of mechanical calculations, and design a reliable plan of solution. Therefore no sophisticated programming tricks are used; all commands are presented in an easily understandable form. A side-effect of use of computer software is the necessity to formulate essential parts of your solutions in a formalized manner so that a computer can accept them.

Calculations and final results, although most of them are given in the Appendix, certainly do not constitute the main purpose of the problem collection. The problems intentionally do not contain hyperlinks to the results. A self-reliant check of correctness is much more valuable then a consultation of the Appendix. The main purpose is to show how to move around in unfamiliar surroundings (e.g. without a deep understanding of mathematics), and to show how to sharpen mathematical skills. Basic knowledge of calculus is a prerequisite and almost all of the problems include references to elementary and analytic geometry, number theory, group theory, linear algebra and matrix theory, theory of equations, including difference and differential equations, advanced calculus, numerical methods of calculus and linear algebra or other topics. Also basic knowledge of syntax and semantics of MATHEMATICA® and/or Maple® is assumed.

The traditional sequence of topics in learning mathematics is not respected: when studying sequences, special knowledge of integrals might be necessary; when dealing with integrals we may need topics from approximation theory, numerical analysis or computer science; linear algebra is often applied in solving problems of analysis. Last but not least, applied problems rarely point towards a predefined part of mathematics; on the contrary, the mathematical tools have to be decided upon after analysis of the problem. The book does not cover the common curriculum of calculus; the grouping of chapters into Concepts, Tools and Applications rather suggests the purpose of different parts of calculus. The Concepts part include mappings, infinite sequences, periodicity, the Tools part consists of finite sums, inequalities and collocation methods and in Applications we include arcs and curves, moments and centers of gravity, maximal and minimal values and some miscellaneous problems.

The main purpose of the book is twofold: (1) to deepen understanding of mathematical concepts and methods, and (2) to show how advanced computer software can help to enhance mathematical thinking.

How To Use This Book

The book is neither a textbook nor an exercise book, although it contains definitions, theorems and examples commonly belonging to calculus. It is a collection of problems, some of them rather difficult, organized in a specific way and using computer software as much as possible to raise the level of mathematical thinking.

This collection of problems may be used by any student wishing and willing to work towards better understanding of mathematics and its applications. It will hopefully help teachers of mathematics to find various ways how to explain basic mathematical concepts and how to motivate their students. Our goal will be achieved if it helps the reader to create, formulate and solve his own simple problems leading to answers to those considered difficult.

The ordering and numbering in the book is mostly used to denote a topic. The reader may freely move around and pick up problems at his will, at random or after consulting the index or the contents (TOC.nb in MATHEMATICA® or 00_ Contents.mw in Maple®). Hyperlinks will lead him towards a suitable starting point. A more systematic way is to follow the Suggestions given at the beginning of each chapter. The suggestions include

some pointers toward topics usually not included in basic calculus problems. It is possible just to follow the interconnection of problems, to find some stimuli in historical remarks (printed in a different font after some of the examples) and to discover how advanced software can be included in the process of understanding mathematical concepts.

At the beginning of each chapter a short summary is given. It describes what type of topics can be learned and why they are important. It also lists simple commands in MATHEMATICA® and Maple®, which can be useful in dealing with the topics included. In the ESM these commands are printed in red; pressing the F1 or F2 key in MATHEMATICA® and Maple®, respectively, evokes the corresponding help file. Here the syntax and semantics are explained and examples of its use are given. Most of the chapters contain at the end references to textbooks or monographs containing information on topics outside basic calculus, which are dealt with in the chapter. References at the end of the book list mainly problem books, sources which inspired the authors or where some of the problems were taken from.

The Printed Book and the ESM

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The book can be used without a computer anywhere, although then the advantages of its structure are that way partly lost. The best way to use it is to sit in front of a powerful computer equipped with MATHEMATICA® or Maple® and use the files in ESM, which contains the text of the book structured according the rules of this software. Interactive work on selected problems is the most effective use. From the file named Contents.nb or Contents.mw chose the chapter by hyperlink. In such an environment the user may open his own MATHEMATICA® notebook or Maple® worksheet, keep records of his attempts to find a solution, and save partial and final results. Also all the hyperlinks and help files are readily available.

A basic short introduction to MATHEMATICA® or Maple® can be found at www.wolfram.com/broadcast/screencasts/handsonstart/ and to Maple® at www.maplesoft.com. If the MATHEMATICA® package is not installed on your computer you can freely download the MATHEMATICAPlayer from www.wolfram.com. This program makes all the notebooks readable and all the hyperlinks function but no other use of MATHEMATICA® is allowed. Also the help files are not accessible. The files on the ESM are text files, but reading them directly is not possible since they contain text formatted in a very specific way.

Graphics and Symbols

Various types of text as described above are given in the ESM in specific fonts and colors. Mathematical symbols in MATHEMATICA® and/or in Maple® often differ from those common in textbooks, monographs and journals. In the printed text we adhere to the com-

mon mathematical notation. For comparison some of the differences and also some explanations are given in the following table.

Symbol	Name and definition	MATHEMATICA®	Maple®
$\mathbf{C}, \mathbf{R}, \mathbf{Q}$	set of complex, real, rational numbers	Complex, Real	Complex
$\mathbf{Z}, \mathbf{N}$	set of integers, natural numbers	Integer	
(a,b)	open interval of reals, $\{x : a < x < b\}$		
$[a, b]$	closed interval of reals, $\{x : a \leq x \leq b\}$		
$\lfloor x \rfloor$	$\lfloor x \rfloor = n \Leftrightarrow n \leq x < n+1, n \in \mathbf{N}$	Floor[x]	floor(x)
n(mod m)	remainder after dividing n by m	Mod[n,m]	mod
n!!	product of integers $\leq n$ of equal parity	n!!	
i	imaginary unit, $i^2 = -1$	I	I
iff	if and only of		
$\bar{a}$	complex conjugate of $a \in \mathbf{C}$	Conjugate[a]	
f_{-1}	the function inverse to f	Inverse[f]	

The Notebooks and Worksheets were written in MATHEMATICA® Version 7.0 and Maple® 14 respectively. Using earlier versions of the software may cause minor difficulties, mainly with commands not supported.

Concluding Remarks

The basic idea of this project—a structured set of interconnected problems—needs verification by students and teachers for their choice, structure, estimated level of difficulty, help topics etc. We are eagerly waiting for comments and further suggestions from users. An easy contact to the authors is through the websites and e-mail at gregorj@math.feld.cvut.cz or tiser@math.feld.cvut.cz and for the Maple® worksheets nemecek@math.feld.cvut.cz.

We hope that our approach will contribute to better understanding of calculus for applications in science and engineering and also help to raise the level of mathematical education, which is now enabled by advances in computer science. A similar approach, if successful, could be applied to other parts of mathematics usually included in mathematical curricula.

Acknowledgements

Our thanks are due to our colleague Aleš Němeček for his valuable contribution. His comments and mainly his work in transposing all the chapters, originally written in MATHEMATICA®, to Maple® worksheets and filling in the Maple® version of all computation commands in the printed version was laborious and demanded a high level of understanding. His effort considerably contributes, we hope, to widen the range of potential users of this book.

We also want to thank Karen Borthwick, Lynn Brandon and Lauren Stoney, all with Springer Verlag, and Lyn Imeson for their help during all stages of preparing this publication.

1 Mappings, Composite and Inverse Functions

Mapping is a basic and rather general concept of analysis. In calculus special mappings called functions are mostly investigated. Special attention in analysis is devoted to functions $f : \mathbf{R} \to \mathbf{R}$ called real functions of one real variable. Some of these functions used to be called elementary functions for historical reasons, others are known as special functions. More general mappings often use methods and concepts of one-dimensional analysis. Geometrical investigation of mappings deals with invariants of mapping such as distance-preserving, angle-preserving mappings, fixed points of mappings etc. Various mappings $\mathbf{R}^3 \to \mathbf{R}^2$ have important applications in visualizing spatial objects on planar media. Mappings $\mathbf{C} \to \mathbf{C}$ described by complex functions of a complex variable have found many application in geometry, field theory and other parts of mathematics.

For functions the concepts of continuity, limits and derivatives of functions are assumed to be known. In this chapter you will learn more about composition, one-to-one mappings, the concept of inverse mapping and others. It is important to note that both in principle and in the syntax of MATHEMATICA® and Maple® the function f, its value $f(x)$ at the point x and the equation $y = f(x)$ used for visualization of f have to be carefully distinguished.
The solution of almost all of the problems here can be made easier when using software packages comparable to S. Wolfram's MATHEMATICA® or Maple®, although only very few of them can be solved directly. The user is strongly advised to learn the syntax and semantics of some of the commands. In this chapter the following commands (and related ones) may help:

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In MATHEMATICA®:
`Plot, ParametricPlot, Integrate, NIntegrate, FindRoot, Solve, EllipticF, ArithmeticGeometricMean, Eigenvalues`
In Maple®:
`plot, int, evalf(Int), solve, roots, EllipticF, LinearAlgebra[Eigenvalues], arithmetic-geometric mean.`
Additional information on basic concepts can also be found on the Internet, e.g. at http://www.maplesoft.com/applications/
In most of the examples numerical experiments may give a starting point of reasoning or verify initial conjectures.
Additional information on basic concepts can also be found on the Internet, e.g. at www.mathworld.wolfram.com or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://eom.springer.de.

Suggestions

Mainly composite and inverse mappings defined by real functions of real variables are considered. Also some examples involving mappings defined by complex functions of a complex variable are included and some basic problems with various sets of mappings are formulated. Accordingly,

– problems M 02, M 03, further M 41, M 43 and also M 16, M 18 may serve as introductory ones,
– M 07, M 08, M 50 and M 43, M 44 and M 39, M 40 are examples of intermediate difficulty,
– M 06, M 13, M 25 and M 49 and M 19 are included so as to give an outlook on more general and applied ideas.

All readers are strongly encouraged to modify, generalize or simplify the formulated problems, to find alternative formulations and formulate, solve their own examples and compare the context of these problems to the given ones.

Problems

M 01 • [M] ↓ M 12

Denoting the inverse of the one-variable function f by f_{-1} it holds that $f(f_{-1}(x)) = x$ and $f_{-1}(f(x)) = x$ whenever x belongs to the domain of the composite functions. Explain the results obtained with the following commands.
In MATHEMATICA®:

```
  Plot  [Cos  [ArcCos  [x]],{x,-2,2}]; \\
Plot  [ArcCos  [Cos  [x]],{x,-2,2}].
```

In Maple®:

```
  plot(cos(arccos(x)), x = -2 .. 2);
plot(arccos(cos(x)), x = -2 .. 2);
```

M 02 •

Let f, g be strictly increasing functions on **R**. Formulate the conditions which allow you to prove the following statements.

1. The inverse of f is increasing.
2. The composite function f of g and/or g of f is increasing on **R**.
3. $f + g$, fg and f/g are increasing on **R**.

M 03 • ↓ M 04

A strictly increasing and continuous real function of one real variable has an inverse. Which of the two assumptions can be omitted so the inverse function still exists?

1

M 04 • [M] ↑ **M 03**

Find the inverse function f_{-1} of f defined below and verify, moreover, that it satisfies $f_{-1} = f$.
In MATHEMATICA®:

```
Plot[If[0 <= x && x < 1, x, 3 - x], {x, 0, 2}]
```

In Maple®:

```
f:=piecewise(0 <= x and x < 1, x, 3-x);
plot(f, x=0..5, discont=true);
```

Construct similar examples of noncontinuous and non-monotonic functions whose inverses exist.

M 05 • ↑ **M 01**

Can you express the derivative of a function which is inverse to f in terms of f and f'?

M 06 •••

Construct a one-to-one mapping $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ of the set of points in the plane with positive and integer coordinates onto the set of positive integers. Find its inverse.

The existence of such a mapping serves as the basis of a proof that the set of all n-tuples of positive integers is countable, hence that the set of rational numbers is countable etc.

M 07 •• ↓ **M 08** ↓ **M 09**

Let f be defined on a subset of $\mathbf{R}$, which is symmetric about the origin. Prove that such a function can be uniquely decomposed into a sum of an even and odd function. Find such a decomposition of the functions $\exp(x)$ and $\exp(\mathbf{i}\,x)$. Derive basic formulae for hyperbolic functions of sums, multiple arguments and similar ones. Compare them with analogous formulae of trigonometry.

Hint: Note that the function $(f(x) + f(-x))/2$ is an even function.

M 08 •• [M] ↑ M 07 ↓ M 28 ↓ M 31

Consider the part of the hyperbola $y^2 - x^2 = 1$ corresponding to $y > 0$. Find the area A of the region lying between the cone formed by two beams starting at the origin, symmetric around the y-axis and below the hyperbola. Express the x coordinate of the endpoint of this arc in terms of the area A.

The solution of this problem enables to find a geometric definition of hyperbolic functions similar to the geometric definition of trigonometric functions based on the area of a circular sector and the length of its tangent. This definition may give a reason why the even and odd parts of the exponential function are called hyperbolic functions.

M 09 • ↑ M 07

Show that the function $f(x) = x/2 + x/(\exp x - 1)$ can be defined as continuous for all real values of x and prove that it is an even function.

M 10 • ↑ M 03 ↑ M 07

Let f be defined on a subset of $\mathbf{R}$ which is symmetric about the origin and let its inverse f_{-1} exist.

1. Prove that f cannot be even. Moreover, f_{-1} is odd iff f is odd.
2. Decide whether the two (unique) decompositions $f = \mathrm{Ev}(f) + \mathrm{Od}(f)$ and $f_{-1} = \mathrm{Ev}(f_{-1}) + \mathrm{Od}(f_{-1})$ imply that $\mathrm{Od}(f_{-1})$ is an inverse of $\mathrm{Od}(f)$, where $\mathrm{Ev}(f)$ and $\mathrm{Od}(f)$ means $(f(x) + f(-x))/2$ and $(f(x) - f(-x))/2$, respectively.

M 11 •• X ↑ M 03 ↑ M 04 ↑ M 07 ↓ M 48

Denote by f_{-1} the inverse of $f : [a, b] \to \mathbf{R}$. Construct examples of functions which are inverse to themselves, i.e. $f = f_{-1}$.
Prove that a continuous function f cannot satisfy any of the following conditions on a subinterval of their common domain: $f = -f_{-1}$, $f = 1/f_{-1}$, $f = -1/f_{-1}$.

Hint: Consider monotonicity of the functions involved.

The equation $f(f(x)) = x$ for the unknown function f is known as the Babbage equation.

M 12 •• **Q 07** **↑ M 01**

The function $\cos(n \arccos(x))$ for nonnegative integer values of n and $x \in [-1, 1]$ is a polynomial of degree n. Prove this statement by induction. Prove also that these polynomials have only real and simple zeros and that all their zeros belong to the interval $(-1, 1)$.

Hint: Find $\cos((n+2) \arccos(x)) + \cos(n \arccos(x))$.

M 13 •• **↑ M 12**

Consider a similar approach as in M 12 with the functions $\sin(n \arcsin(x))$, $n > 1$ or $\sin((n+1) \arccos(x)) / \sin(\arccos(x))$ on $[-1, 1]$. Which of the statements above remain true?

The polynomials in M 12, M 13 are called Čebyšev (Tschebyscheff or Chebyshev) polynomials. They play an important role in the theory of approximation.
See the commands `ChebyshevT[k,z], ChebyshevU[k,z]`.

M 14 •• **↓ M 15**

Prove that

$$\arccos x = \arcsin \sqrt{1-x^2} \text{ for } 0 < x < 1.$$

Similarly, express one inverse trigonometric function in terms of the others. Use the formula for $\sin(x+y)$ and the previous result to prove that

$$\arcsin x + \arcsin y = \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

for $xy \leq 0$ or $x^2 + y^2 \leq 1$, and

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}, \quad xy < 1.$$

M 15 • ↑ M12 ↑ M 13 ↑ M 14

Prove that $\sin(2n \arctan x)$ and $\cos(2n \arctan x)$ are rational functions. Find the zeros of these functions; are they simple? What can be said about the zeros of the denominators?

Hint: Prove that for $x > 0$ there is

$$\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}} = \arccos \frac{1}{\sqrt{1+x^2}}$$

and follow the pattern of M 12.
Alternatively, use the following identity for $y = \tan(x)$:

$$\exp(2\mathbf{i}\,x) = \frac{1+\mathbf{i}\,y}{1-\mathbf{i}\,y}.$$

M 16 • ↓ M 17

Consider a linear mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$. Prove that the image of a parallelogram is again a parallelogram.

Hint: A parallelogram is a set of points c given as $c = sa + tb$, where $a, b \in \mathbf{R}^2$, and $s, t \in [0, 1]$.

M 17 • **[M]** ↓ M 39 ↓ M 40

Find a linear mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ which maps the unit square to the parallelogram with vertices $(0, 0)$, $(1, 1)$, $(3, -5)$, $(4, -4)$. Find the Jacobian of the mapping and its geometrical meaning.

M 18 •

Show that the mapping $\phi(r, \alpha) = (r \cos\alpha, r \sin\alpha)$ maps a rectangle in the (r, α) plane onto a region with boundaries given by two concentric circles and two rays from the origin. Compare the ratio of the areas of the two regions with the Jacobian of the mapping.

M 19 ●● ↓ M 20 ↓ M 21

Find the mapping $f : \mathbf{R}^3 \to \mathbf{R}^2$. defined by the following construction: Each point (x, y, z) is orthogonally projected into a plane passing through the origin with a fixed normal vector (a, b, c). Work out a program, which enables to experiment by changing the parameters a, b, c and show how rotation of a three-dimensional object can be visualized.

This construction is called axonometry. It is the most common way of visualizing spatial objects on a plane.

M 20 ●● A 19 ↑ M 19 ↓ M 21

1. Describe the mapping f which maps the surface of a unit sphere to a cylinder wrapped around the sphere so that each point on the sphere and the corresponding point on the cylinder are on a beam with the endpoint at the center of the sphere.
2. Describe the mapping f which maps the surface of a unit sphere to a cylinder wrapped around the sphere so that the mapping preserves the angle of any two curves on the sphere.

The mapping of the globe in the second case was invented around 1550 by Gerhardus Mercator to help sailors to determine courses, since a course with constant heading is a straight line on a Mercator map. Such mapping can be generalized when considering an ellipsoid as a more adequate model of the earth surface. The mapping in the first case is often confused with Mercator's mapping.

M 21 ●● ↑ M 19 ↑ M 20

Find the mapping f from the unit sphere to $\mathbf{R}^2$ defined by the following construction: A sphere of radius 1 (say $\xi^2 + \eta^2 + (\zeta - 1)^2 = 1$), and its tangent plane $\zeta = 0$ are considered. The points of the sphere are projected onto its tangent plane from a point of the sphere 'opposite' to the tangent point $(0, 0, 0)$, i.e. from the point $(0, 0, 2)$. Is such a mapping a bijection? Prove that such a mapping is angle-preserving.

This mapping is called the stereographic projection. It is used as a model of the complex plane (the so-called Gauss plane), to illustrate the role of the complex number ∞.

M 22 • ↑ **M 05** ↓ **M 23**

Let f be a continuous function on the interval $(0,a)$ with $f(0)=0$ and $0<f'(x)\le M$ on this interval. Prove that

$$\int_0^a (f(x))^n\,dx \ge \frac{1}{M}\frac{f^{n+1}(a)}{n+1}.$$

Find an analogous result with the assumption $0<f'(x)\le M$ replaced by $f'(x)\ge M>0$.

Hint: Use the substitution $f(x)=y$.

M 23 •• ↑ **M 05** ↑ **M 22**

Let f be a continuous function on the interval $(0,1)$ with $f(0)=0$ and $0<f'(x)\le 1$ on this interval. Prove that

$$\left(\int_0^1 f(x)\,dx\right)^2 \ge \int_0^1 f^3(x)\,dx,$$

where equality occurs iff $f(x)=x$ or $f(x)=0$.

Hint: Replace the inequality by the one involving the inverse function to f, or define

$$F(t)=\left(\int_0^t f(x)\,dx\right)^2-\int_0^t f^3(x)\,dx$$

and prove that F is increasing, e.g. by noticing that $F'(0)=0$ and F' is increasing: together with $F(0)=0$ this leads to the desired result.

M 24 • ↑ **M 11**

Let f be a strictly increasing and continuous function $f:(a,b)\to\mathbf{R}$ and let f_{-1} be its inverse. Prove that

$$\int_a^b f(x)\,dx+\int_{f(a)}^{f(b)} f_{-1}(x)\,dx = bf(b)-af(a).$$

Formulate an analogous conclusion for strictly decreasing functions. Use this result to evaluate some integrals containing inverse functions e.g. $\int_1^2 \arcsin(1/x)\,dx$.

1

Hint: Integrate the expression $(xf(x))'$ or consider geometrical interpretation of the two integrals.

M 25 • ↓ **PM 25** ↑ **M 02** ↑ **M 03** ↑ **M 05** ↓ **M 27**

Use the methods of previous examples M 02 , M 03 and M 05 to investigate the functions (pretending that you don't know them explicitly)

$$s(x) = \int_0^x \frac{1}{\sqrt{1-y^2}}\,dy, \quad -1 \le x \le 1$$

$$ss(x) = \int_0^x \frac{1}{\sqrt{1+y^2}}\,dy, \quad x \in \mathbf{R}$$

and their inverse functions.

M 26 •• ↓ **PM 25** ↑ **M 02** ↑ **M 03** ↑ **M 05** ↓ **M 27**

Repeat the previous example with the function

$$sl(x) = \int_0^x \frac{1}{\sqrt{1-y^4}}\,dy$$

and its inverse.

Hint:
```
Plot[NIntegrate[1/Sqrt[1 - y^4], {y, 0, x}],
  {x, 0, 1}]
```
in MATHEMATICA® or
```
plot(evalf(Int(1/sqrt(1-x^4), x = 0 .. t)),
  t = 0 .. 1)
```
in Maple®.

M 27 •• ↓ **PM 25**

For a fixed value of k with $k^2 < 1$ prove that

$$\int_0^{\sin a} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx = \int_0^a \frac{1}{\sqrt{1-k^2\sin^2 x}}\,dx.$$

Denote this integral by $F(a,k)$ and find that this function (for a fixed value of k) is an increasing unbounded function of a, it is concave and convex in intervals $(2n\pi, (2n+1)\pi)$, $((2n+1)\pi, (2n+2)\pi)$, respectively , for any integer n. Similarly, find basic properties of the integral for $a = \pi/2$ as function of k.

The function $F(a,k)$ for a fixed k is called the incomplete elliptic integral of the first kind and $F(\pi/2,k) = K(k)$ is called the complete elliptic integral of the first kind. In MATHEMATICA® they are calculated as `EllipticF[a,k] and EllipticK[k]`,
in Maple® as `EllipticF(a,k) and EllipticK(k)`

M 28 •• ↑ M 08 ↑ M 27

Express $\mathrm{sl}(x)$ of M 26 in terms of the incomplete elliptic integral $F(a,k)$. Find a geometric interpretation of the function $\mathrm{sl}(x)$ and its inverse by expressing the arclength of the curve $r^2 = \sin(2t)$ (lemniscate of Bernoulli) in terms of the polar angle t.

Hint: When modifying $1 - y^2$ use $1 + y^2 = 2(1 - \frac{1-y^2}{2})$.
For curves in polar coordinates (r,t) the element of the arclength is

$$ds = \sqrt{r^2 + \left(\frac{dr}{dt}\right)^2}\, dt = \sqrt{r^2\left(\frac{dt}{dr}\right)^2 + 1}\, dr.$$

M 29 •• ↑ M 27

Physical considerations for the mathematical pendulum of length l and initial angle α_0 allow to express the time instant t when the pendulum assumes the angle α as

$$t = \frac{1}{2}\sqrt{\frac{l}{g}} \int_{\alpha}^{\alpha_0} \frac{1}{\sqrt{\sin^2(\frac{\alpha_0}{2}) - \sin^2(\frac{\alpha}{2})}}\, d\alpha.$$

Express this time in terms of $F(a,k)$ or in terms of the function in Example M 27. For a pendulum of unit length find the time when the initial angle is halved. Compare your answer with that obtained from the linear model of the mathematical pendulum.

M 30 ••• I 63 ↓ M 33

For $a_1 = \frac{a+b}{2}$, $b_1 = \sqrt{ab}$, $0 < b < a$ prove that

$$\int_0^{\pi/2} \frac{1}{\sqrt{a^2\cos^2 x + b^2\sin^2 x}}\, dx = \int_0^{\pi/2} \frac{1}{\sqrt{a_1^2\cos^2 x + b_1^2\sin^2 x}}\, dx.$$

1

This result is known as the Gauss formula and it yields (via the arithmetic-geometric mean) one of the most effective ways to evaluate elliptic integrals.

Hint: Work out the following substitution in the left-hand side integral

$$\sin u = \frac{2a\sin x}{(a+b)+(a-b)\sin^2 x},$$

where $0 \le x \le \pi/2$ corresponds to the same interval for the new variable u. Take into account (verify it!) that

$$\cos u = \frac{\sqrt{(a+b)^2-(a-b)^2\sin^2 x}}{(a+b)+(a-b)\sin^2 x}\cos x.$$

M 31 • ↑ **M 28**

Find the arclength of the ellipse given as $x = a\cos t,\ y = b\sin t$. By appropriate choice of the half-axes give a geometric interpretation of the function

$$E(t,k) = \int_0^t \sqrt{1-k^2\sin^2 x}\,dx, \quad |k| < 1,$$

and its inverse.

The function $E(t,k)$ for a fixed k is called the incomplete elliptic integral of the second kind and $E(\pi/2,k) = E(k)$ is called the complete elliptic integral of the second kind.

M 32 • ↑ **M 27** ↑ **M 30**

Express the integral in M 30 in terms of $K(k)$ of M 27.

M 33 •• **I 63** ↑ **M 30**

Form two sequences $a(n), b(n)$ as follows:

$$a(n+1) = (a(n)+b(n))/2, \qquad b(n+1) = \sqrt{a(n)b(n)}$$

with $a(1) = a$, $b(1) = b$, $0 < b < a$.
Prove that both sequences have limits and that these limits are equal. This limit is called the arithmetic-geometric mean, shortly (AGM) of a and b.

Hint: You may try to calculate a few terms of these sequences but, of course, this proves nothing. Note that $a(n) > b(n)$ for all $n > 0$.

M 34 • **I 63** **↑ M 27 ↑ M 30 ↑ M 32**

Can you prove that the arithmetic-geometric mean as it was defined in the previous example is proportional to the reciprocal value of the integral on the left-hand side of the equality in Example M 27?

M 35 •

Prove that the mapping $f(x) = ax^3, a \neq 0$ is injective and surjective if considered as $f : \mathbf{R} \to \mathbf{R}$ but considered as $f : \mathbf{C} \to \mathbf{C}$ it is surjective but not injective.

M 38 •• [M]

For a given function f approximate its inverse by interpolation, i.e. find a polynomial of given degree n assuming the values x_k at points $f(x_k)$, $k = 1, 2, \ldots, n+1$. Use the approximation to solve the equation $\sin x = ax$, $0 < a < 1$, for the unknown value x.

Hint: The command `InterpolatingPolynomial` in MATHEMATICA® or `CurveFitting[PolynomialInterpolation](xdata,ydata,v,opts)` in Maple® can be used. For $n = 1$ this method is known as regula falsi, for $n = 2$ it is called the method of Brent.

M 39 • ↑ M 17 ↓ M 40

Consider the set of mappings $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined as follows:

$$\xi = a_{11}x + a_{12}y, \qquad \eta = a_{21}x + a_{22}y,$$

with $\det\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) \neq 0$.
Prove that this set of mappings forms a noncommutative group with composition as the group operation. How would you express this composition in terms of a_{ik} and b_{ik} respectively? Compare your result with the well-known operation of matrix multiplication. Find the neutral (or unit) element of this group. Find the representation of an inverse element in terms of a_{ik}.

M 40 • ↑ M 17 ↑ M 39

Find conditions for the coefficients a_{ik} for the mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$\xi = a_{11}x + a_{12}y, \qquad \eta = a_{21}x + a_{22}y,$$

to be distance preserving. Give a geometric interpretation of your result.

M 41 • ↑ M 21 ↑ M 40 ↓ M 44

Find whether the mapping $f : \mathbf{C} \to \mathbf{C}$ defined as $f(z) = az$, $a \in \mathbf{C}$ is distance and angle preserving.

M 43 • ↑ M 40 ↓ M 44

Find all distance preserving holomorphic mappings $f : \mathbf{C} \to \mathbf{C}$.

Hint: For $|f(z_1) - f(z_2)| = |z_1 - z_2|$ consider the existence of the derivative and the Cauchy–Riemann equations for f.

M 44 •• **A 17** ↑ **M 41**

Prove that a function holomorphic in a domain $D \subset \mathbf{C}$ represents an angle preserving mapping.

Hint: Consider two arcs $z_i(t)$ with a common point at $t = t_0$, their tangent vectors and tangent vectors of their images under a holomorphic mapping f.

Mapping with holomorphic functions, the so-called conformal mapping, found important applications in planar problems of hydro- and aero-dynamics and in modeling of electrostatic and other fields.

M 45 •• **[M]** ↓ **M 48**

For a given polynomial P_n of degree n the function

$$Q(x) = (1+x)^n P_n\left(\frac{1-x}{1+x}\right)$$

is also a polynomial of degree n. Prove this statement and try to find an algorithm which would express the coefficients of Q in terms of those in P.

M 46 • ↑ **M 45** ↓ **M 48**

If the polynomial P has a (complex) zero at point z_0 with $|z_0| < 1$ then the polynomial Q in M 45 has a zero w_0 with $\operatorname{Re} w_0 > 0$ and conversely. Find a proof.

Hint: You should consider mappings in the complex plane as in M 48.

M 47 •

Consider the mappings of the form $h(x) = \frac{ax+b}{cx+d}$, $ad - bc \neq 0$ and try to find its fixed points, if they exist. Distinguish cases $x \in \mathbf{R}$ and $x \in \mathbf{C}$.

1

M 48 • ↑ **M 45**

In the complex plane consider the mapping $w = \frac{z-1}{z+1}$.
Prove that it maps the right half-plane onto the unit circle and also the upper half-plane onto itself.

M 49 •• ↑ **M 40** ↑ **M 41**↑ **M 48**

In the complex plane consider the mapping

$$w = \frac{z-\alpha}{1-\overline{\alpha}z},$$

with the complex number α satisfying $|\alpha| < 1$.
Prove that it maps the unit circle onto itself. Try to prove that the set of these mappings form a group with respect to composition. If not, modify the mapping so as it forms a group.

Hint: Use the command `ParametricPlot` in MATHEMATICA® or the `plot` command in Maple®.

The mappings in M 48, M 49 may be taken as concepts on which Poincare's model of hyperbolic (Lobachevski) geometry can be visualized. Bolyai and Lobachevski indirectly proved the independence of the famous fifth postulate of Euclid: namely the fact that in the (Euclidean) plane a straight line has one and only one parallel straight line containing a point off the given line. They showed the system of axioms, where this fifth postulate is replaced by its opposite and yields a meaningful geometry.

M 50 •

Let F be a real-valued function defined for all real x except for $x = 0$ and $x = 1$ and satisfying the functional equation $F(x) + F((x-1)/x) = 1 + x$. Find all functions F satisfying these conditions.

Hint: In the equation put $\frac{x-1}{x}$ for x twice in succession and find $F(x)$.

M 51 • ↑ **M 07**

Find examples of functions F, G, H satisfying the equations:

1. $F(x) = F(1-x)$,
2. $G(x) = G(a-x)$, $a > 0$,
3. $H(x) = H(1/x)$.

M 52 •

Find all holomorphic functions F satisfying the equation

$$F(x+y) = F(x)F(y) \quad x, y \in \mathbf{C}.$$

Hint: Consider F in the form of a Taylor series with unknown coefficients.

Supplementary Material

Definitions

Let $\varphi : A \to B$ be a map. A is called its domain $D(\varphi)$, B is called its range $R(\varphi)$. The map is called

– *injective* or one-to-one, if $\varphi(a_1) = \varphi(a_2)$ implies $a_1 = a_2$ for all $a_1, a_2 \in A$, i.e. no two different elements of A ever have the same image in B under φ,
– *surjective* or onto if $\varphi(A) = B$, i.e. for each $b \in B$ there exists an $a \in A$ with $\varphi(a) = b$,
– *bijective* if it is both injective and surjective.

A mapping φ is called *linear* over T if $\varphi(\lambda x) = \lambda\varphi(x)$ for all $\lambda \in T$ and $\varphi(x+y) = \varphi(x) + \varphi(y)$.

Let $\varphi : A \to B$, $\psi : B \to C$ be maps. Then the composition $\psi \circ \varphi$ of φ and ψ is the map $\psi \circ \varphi : A \to C$ defined by $\psi \circ \varphi(a) = \psi(\varphi(a))$ for all $a \in A$.

Let $\varphi : A \to \mathbf{R}^n$, $A \subset \mathbf{R}^n$ and let
$\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))$, $x \in \mathbf{R}^n$.
If $\frac{\partial \varphi_k(x)}{\partial x_i}$ exists for all $1 \le i, k \le n$ then the determinant of the matrix J with elements $\frac{\partial \varphi_k(x)}{\partial x_i}$ is called the Jacobian of the mapping φ.

A complex function f of a complex variable $z = x + \mathbf{i}\, y$, $\mathbf{i}^2 = -1$ is called holomorphic (or analytic or regular) in a domain D if it is differentiable at any $z \in D$, i.e. if for any $z \in D$ the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all $z \in D$.

A group is defined as a nonempty set S of elements with an operation $\circ$ for which the following axioms are fulfilled:

- for any $a, b \in S$ there is $a \circ b \in S$ and $(a \circ b) \circ c = a \circ (b \circ c)$, there is an element $e \in S$ (called the unit element) such that $a \circ e = e \circ a = a$,
- for each $a \in S$ there exists an inverse element a^{-1} such that $a^{-1} \circ a = a \circ a^{-1} = e$,

if, in addition, $a \circ b = b \circ a$ then the group is called commutative.

If f is a mapping, then any point x which satisfies the equation $f(x) = x$ is called its fixed point.

Theorems

A

A complex function f of a complex variable $z = x + \mathbf{i}\, y$, $\mathbf{i}^2 = -1$ with $f(z) = u(x, y) + \mathbf{i}\, v(x, y)$, where u, v are real functions of two real variables, is holomorphic in a domain D if u and v are continuously differentiable functions satisfying the so-called Cauchy–Riemann equations for all $z \in D$

$$\partial_x u(x, y) = \partial_y v(x, y),$$
$$\partial_y u(x, y) = -\partial_x v(x, y).$$

Plans of Solutions

PM 25

1. Find the sets where s and ss are defined and find the range of both functions.
2. Taking derivatives decide on monotonicity, existence of the inverses, convexity of s and ss.
3. Use numerical integration to verify your conclusions and obtain some information on the graph of the inverses.
4. In cases of s and ss the whole procedure can be verified finding the four elementary functions involved.
5. For the function sl apply the first three steps above.

Further References

Examples M 25 – M 32 are closely related to the theory of elliptic functions. Many textbooks and handbooks contain relevant information, see e.g.

Whittaker E.T, Watson G.N., A Course of Modern Analysis, Cambridge University Press, 1927.

Hurwitz A., Courant R., Vorlesungen uber Allgemeine Funktionentheorie und Elliptische Funktionen, Springer, Berlin, 1964, and many others.

Examples M 21, M 35, M 41, M 43, M 44, M 47 – M 49 are taken from the theory of complex functions of a complex variable. Additional information can be found in many textbooks, e.g. in

Conway J.B., Functions of One Complex Variable, Springer, New York, 1973.

2 Infinite Sequences

Infinite sequences and their limits are basic concepts in mathematical analysis with widely spread applications in numerical mathematics, theory of equations and many other parts of mathematics. The original concept can be traced back to the 17th century. Development of functional analysis and basics of modern mathematics at the beginning of the 20th century showed its strength and significance.

In this chapter you will learn how and where infinite sequences originated, how they can be described and defined, and what kind of questions and problems can be posed and solved. By solving the easy problems you will learn how to approach more involved ones and finally you will be ready to solve the most difficult tasks formulated in this section.

The solution of almost all of the problems here can be made easier when using software packages comparable to S. Wolfram's MATHEMATICA®, or Maple® of Waterloo Maple Inc., although only very few of them can be solved directly. The user is strongly advised to learn the syntax and semantics of the corresponding commands. In this chapter the following commands (and related ones) may help: In MATHEMATICA®:

```
Limit, ListPlot, AppendTo, Do, Reduce
ContinuedFractions, EllipticK, Floor, Ceiling,
Integrate, ArithmeticGeometricMean, Fibonacci,...
```

Additional help can be found in packages

```
DiscreteMath`*.*`.
```

J. Gregor, J. Tišer, *Discovering Mathematics*,
DOI 10.1007/978-0-85729-064-9_3,

2

In Maple®:
`evalf, seq, floor, ceil, Reduce, assign, plot, int, listplot, numtheory[cfrac],combinat[fibonacci], EllipticK, GaussAGM`

Additional help can be found after entering `?index` and pressing `Enter`

In most of the examples numerical experiments may give a starting point of reasoning or help to verify initial conjectures. Explanation of mathematical terms and concepts can also be found on the Internet, e.g. at
www.mathworld.wolfram.com
or at http://en.wikipedia.org/wiki/Portal:Mathematics or at
http://eom.springer.de.

Suggestions

- A good start for beginners are examples I 01, I 02, I 06, I 11, I 17, I 25, I 26 and further along the downward arrows.
- Those research inclined will find interesting stimuli in I 15, I 16, I 22, I 24, I 32, I 43, I 52, I 63 and may use the upward arrows to find help.
- Teachers could use I 08, I 10, I 27, I 29, I 33, I 44 to motivate subsequent work.
- All readers are strongly encouraged to modify, generalize or simplify the given problems, to find alternative settings , formulate and solve their own examples and find the context of these problems to the given ones.

Problems

An infinite sequence can be described by a list of several of its first terms, like $a_1, a_2, a_3, \ldots$. Such a description is not sufficient since it does not define the sequence uniquely. Except for a few special cases it is not used in mathematics; you may find it in riddles in newspapers and journals. The simplest way to define a sequence is a formulation of a rule creating its terms, for instance the sequence a_n given by the rule $a_n = n^3 - n^2$ gives $0, 4, 18, 48, 100, \ldots$. If not stated otherwise it is understood that n are natural numbers, i.e. $n = 1, 2, 3, \ldots$. Such definition of sequences is called explicit. The rule defining a sequence can also give its n-th term as a function of some previous terms, like in $a_n = qa_{n-1}$, $n > 1$. Such definition can be called implicit or recursive. The definition is unique if some initial terms are given, e.g. $a_1 = 1$ in this example. The notation a_n can also be replaced by $a(n)$, since more complicated indexes may cause difficulties in printing. This notation also accords with the notation of functions.

I 01 • [M] ↓ I 11 ↓ I 13 ↓ I 25 ↓ I 34 ↓ I 43

Plot the first few terms of the sequences below and guess their limits, if they exist. Can you prove that your guess is correct?

(i) $x_n = \dfrac{n+1}{n-1}$, (ii) $x_n = n\sin\left(\dfrac{1}{n}\right)$,

(iii) $x_n = \left(1 + \dfrac{(-1)^n}{n}\right)^n$,

(iv) $x_n = (\sqrt{n+1} - \sqrt{n-1})\sqrt{n}$,

(v) $x_n = n^{\frac{1}{n}}$, (vi) $x_n = n \bmod 5$.

I 02 • [M] ↓ I 35 ↓ I 36

Let x_n be a sequence defined recursively by the equations below. Express its n-th term as a function of n, determine whether the sequence converges and if so find the limit of the sequence.

(i) $x_{n+1} = x_n + d, \quad x_0 \in R,\ d \in R$,

(ii) $x_{n+1} = x_n + (-1)^n, \quad x_0 \in R$,

(iii) $x_{n+1} = \lambda x_n, \quad \lambda \in R,\ x_0 \in R$,

(iv) $x_{n+1} = (n+1)x_n, \quad x_0 = 1$,

(v) $x_{n+1} = \sqrt{x_n}, \quad x_0 \geq 0$,

(vi) $x_{n+1} = \alpha^{n+1} x_n, \quad x_0 = 1,\ \alpha \in R$,

(vii) $x_{n+1} = \exp(x_n), \quad x_0 = 0$.

I 03 • ↓ I 05 ↓ I 52

Find, if it exists, $\lim_{n\to\infty} \frac{1}{n}\exp(i\pi n/3)$ and $\lim_{n\to\infty} \exp(i\pi n/3)$ with $i^2 = -1$.

2

I 04 • ↓ **I 05** ↓ **I 06** ↓ **I 07**

Let $a(n) = 1 + (-1)^n$, $b(n) = 1 + (-1)^{n+1}$. Neither of these has a limit, although both are bounded. Form the sequences $c(n) = a(n)b(n)$, $d(n) = a(n) + b(n)$ and find, if possible, $\lim_{n\to\infty} c(n)$ and $\lim_{n\to\infty} d(n)$. Can you modify sequences $a(n), b(n)$ so that they become divergent and unbounded, and still the deduction on $c(n)$ and $d(n)$ holds true?

Hint: Recall the basic theorems.

I 05 • ↑ **I 03**

The limit of the sequence $\{\cos \pi n\}$ does not exist. Take any $\lambda > 0$ and consider the sequence $c_n = \frac{1}{n^\lambda + \cos \pi n}$. Verify that $\lim_{n\to\infty} c(n) = 0$.

Hint: Recall the basic theorems.

I 06 • ↓ **I 07**

Assume that c_n is a bounded sequence and $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 1$. What conclusions can be proved for the sequences

(i) $x_n = a_n \cos \pi n$,
(ii) $y_n = b_n \sin \pi n$,
(iii) $z_n = a_n c_n$,
(iv) $w_n = b_n c_n$?

Hint: Recall the basic theorems.

I 07 • ↑ **I 06**

Let $f = \{f_n\}$, $g = \{g_n\}$ be given sequences such that $\lim_{n\to\infty} f_n g_n = 0$.

(i) What conclusions can be proved for the sequences f and g if they are both convergent?
(ii) What if only one of them is convergent?
(iii) Can it happen that none of them is convergent?

Hint: Recall the basic theorems.

I 08 • • ↓ **PI 08** ↓ **I 09**

Consider a sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = A$ and the sequence $y_n = \frac{x_1+x_2+\cdots+x_n}{n}$. Show that $\lim_{n\to\infty} y_n = A$.

I 09 • ↑ **I 08**

Consider a sequence $\{x_n\}$ of positive numbers with $\lim_{n\to\infty} x_n = A$ and the sequences

$$z_n = \sqrt[n]{x_1 x_2 \ldots x_n}, \qquad w_n = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

Find $\lim_{n\to\infty} z_n$ and $\lim_{n\to\infty} w_n$, if $A > 0$.

Hint: Consider $\log z_n$ and $\frac{1}{w_n}$.

I 10 •• ↓ **I 64**

Consider the unit circle and its circumscribed and inscribed regular polygons with n sides. Denote by $A(n)$ and $a(n)$, $n > 2$, the area of the circumscribed and inscribed polygons, respectively.

(i) Prove that $\lim_{n\to\infty} A(n) = \lim_{n\to\infty} a(n) = \pi$.

(ii) Prove that these quantities satisfy the equations

$$a(2n) = \sqrt{a(n)A(n)}, \qquad A(2n) = \frac{2a(2n)A(n)}{a(2n) + A(n)}$$

and show how to calculate $a(2^m)$ and $A(2^m)$.

(iii) Find $a(6)$ and $A(6)$ without referring to trigonometric functions and evaluate, using the arithmetic and harmonic means above, the first few terms of the sequences $a(3 \cdot 2^k)$ and $A(3 \cdot 2^k)$. Can you find the value of k such that the corresponding term of one of these sequences approximates π by a number in the interval $(223/71, 220/70)$?

2

This approach to finding the area of a disc is traditionally attributed to Archimedes (287–212 BC). It appeared in his treatise 'On the measurement of the circle'. He used the recurrence relation starting with a regular hexagon and doubling the number of sides in each step. The result of his computation was an approximation of π by a number between $310/71$ and $310/70$. It is, however, unlikely that Archimedes was the discoverer of this value of π, since it implicitly appeared earlier in the quadrature of the circle attributed to Dinostratus.

I 11 • **[M]** ↑ **I 01** ↓ **I 12**

Some of the sequences in I 01 converge to the same value. Denote by $N(\varepsilon)$ the smallest value of n for which the inequality $|a(n) - 1| < \varepsilon$ is satisfied. Find this value for a fixed ε in the sequences mentioned in I 01. Is there any difference in the behavior of $N(\varepsilon)$ for various sequences?

Hint: Use the `ListPlot` in MATHEMATICA® or `plots[pointplot]` or `plot` command in Maple®. Give estimates of N for various values of ε.

I 12 • **[M]** ↑ **I 01** ↑ **I 11** ↓ **I 13** ↓ **I 14** ↓ **I 15** ↓ **I 17**

Perhaps you feel that the sequences in I 01 approach their limits with different speed? Would it be possible to measure this 'speed' by comparison with some standard at least in case of finite limits? Can you formalize a concept of 'speed of convergence'? How can this concept be redefined for sequences with infinite limits?

Hint: Use a 'standard' scale of sequences (e.g. n^p, $p \in R$) for comparison of $|a_n - A|$ (see ↓ ord).

I 13 • **[M]** ↑ **I 01**

Find the order of convergence for the convergent sequences in examples I 01 and I 10. What is the largest possible value of p for

(i) $\dfrac{n}{\sqrt{n^2+n}} - 1 = O(n^{-p})$, (ii) $\dfrac{n}{\sqrt{n^2+1}} - 1 = O(n^{-p})$?

I 14 • ↑ I 12 ↑ I 13 ↓ I 15

With given sequences $f(n) = O(n^{-p})$, $g(n) = O(n^{-q})$, $p, q > 0$, find the 'O-estimates' for the following sequences:

(i) $af(n)$, $a \neq 0$,
(ii) $f(n) + g(n)$,
(iii) $f(n)g(n)$.

I 15 •• ↑ I 12 ↑ I 14 ↓ I 16

Assume that the sequence satisfies $x(n) = O(n^{-p})$, $n \to \infty$. Find the order of convergence for the following subsequences

(i) $y(n) = x(2n)$,
(ii) $z(n) = x(n^2)$,
(iii) $w(n) = x(\mu(n))$,

where μ is an increasing function $\mu : \mathbf{N} \to \mathbf{N}$.

I 16 ••• ↓ PI 16 ↑ I 12 ↑ I 14 ↑ I 15

Let x_n be a convergent sequence, $\lim_{n\to\infty} x_n = A$.

(i) Show that for any $p > 0$ there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that the order of its convergence is n^{-p}, i.e. $|y_n - A| = O(n^{-p})$, $n \to \infty$.
(ii) Show that for a given sequence $\{g_n\}$ of positive numbers converging to zero there is a subsequence $\{y_n\}$ of $\{x_n\}$ such that $|y_n - A| = O(g_n)$, $n \to \infty$.

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I 17 • ↓ **I 27** ↓ **I 61** ↓ **I 62**

Consider a sequence x_n satisfying

$$|x_{n+1} - x_n| \leq \lambda |x_n - x_{n-1}|,$$

where $0 < \lambda < 1$.

(i) Is such sequence convergent?
(ii) What is its order of convergence?

Hint: Find that $|x_{n+1} - x_n| < \lambda^n |x_1 - x_0|$ and use the Bolzano–Cauchy theorem.

I 20 • •

Find the limit of the sequence $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$.

Hint: Compare a_n with integral sums of $\int_1^2 \frac{1}{x} dx$ over an equidistant partition of the interval $[1, 2]$.
In another approach take into account that the n-th term of the sequence equals the area below the graph of a piecewise constant function with discontinuities at integer numbers from the interval $(n, 2n)$ and compare it with the integral $\int_n^{2n} \frac{1}{x} dx$.

I 21 •• ↑ **I 08** ↑ **I 09**

Let N be a fixed integer and let $\{a_n\}$ be a sequence of positive real numbers.

(i) Find the limit of the sequence

$$y_n = \sqrt[n]{a_1^n + a_2^n + \cdots + a_N^n}.$$

(ii) Reconsider this limit for

$$z_n = \sqrt[n]{a_1^n + a_2^n + \cdots + a_n^n}.$$

Hint: In (i) factor out $\max a_i$

I 22 ●● ● [M] ↓ PI 22 ↑ I 10 ↓ I 23

Investigate the sequences

$$x(n) = \int_0^{\pi/2} \sin^n x\,dx \quad \text{and} \quad y(n) = \int_0^{\pi/2} \cos^n x\,dx.$$

As one of your results prove that

$$\lim_{n\to\infty} \frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 = \frac{\pi}{2}.$$

The formula bears the name of Wallis. John Wallis (1616–1703), one of the leading English mathematicians before Newton, made his most important contribution in analysis of infinitesimals.

I 23 ●● ↓ PI 23 ↑ I 21 ↓ I 24

Show that the sequence $y(n) = (\int_0^{\pi} |\sin^n x| dx)^{1/n}$ is bounded. Find its limit.

I 24 ● ●● ↓ PI 24 ↑ I 21 ↑ I 23

Let f be a continuous function on the closed and bounded interval (a, b). Investigate the sequence

$$x(n) = \left(\int_a^b |f(x)|^n dx\right)^{1/n}$$

and show that

$$\lim_{n\to\infty} x(n) = \max(|f(x)|, x\varepsilon(a, b)).$$

I 25 ●● ↓ PI 25 ↑ I 01

Prove that the limit of the sequence $x_n = (1 + 1/n)^n$ exists.

2

I 26 • **[M]**

Let a function f be continuous on the closed interval $[a, b]$ and let $f(x) \in [a, b]$ for all $x \in [a, b]$. Choose any $x_0 \in (a, b)$ and construct a broken line passing successively the points $(x_0, f(x_0))$, $(f(x_0), f(x_0))$, $(f(x_0), f(f(x_0)))$, $(f(f(x_0)), f(f(x_0)))$, $(f(f(x_0)), f(f(f(x_0))))$, ... Compare this broken line with the plot of the function f and the function $g(x) = x$.

Hint: Design some experiments for various functions, e.g. $f(x) = \exp(-x)$, $f(x) = \sin x$, and for various 'starting points' x_0.

I 27 •• ↑ **I 17**

Consider the sequence $x(n+1) = \frac{1}{2}(x(n) + \frac{a}{x(n)})$, $a > 0$. Prove that its limit exists for every $x_0 \neq 0$, find it and estimate the speed of convergence towards this limit.

Hint: Assume first that $x(0) > 0$. Find the minimal value of the right-hand side and use it to prove that $x(n)$ is decreasing and $x(n) > \sqrt{a}$ for $n > 0$. What about $x_0 < 0$? Also, prove that $|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$, implying the estimate of the speed of convergence.

I 28 •• ↑ **I 01** ↑ **I 26**

Investigate the limit of the sequence defined by the following recurrence

$$x_{n+1} = \frac{1}{2}(1 + x_n^3)$$

with a given real value x_0.

Hint: Do there exist real values of x_0 such that x_n is a constant sequence? There are three of them and they subdivide the real axis into 4 intervals. With x_0 in one of them, decide whether the sequence x_n is increasing or decreasing by analyzing $x_{n+1} - x_n$.

I 29 ●● ↓ **PI 29** ↓ **I 30** ↓ **I 33**

Prove the following identity

$$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} = \frac{1+\sqrt{5}}{2}.$$

Hint: Find the recurrence formulae for both infinite sequences. Verify that both sequences are convergent using the theorem on bounded and monotone sequences. Find the limits of the two sequences.

The value $\lambda = (1+\sqrt{5})/2$ is called the golden section ratio. This notion goes back at least to the Pythagorean school and appears e.g. in Euclid's Elements. It was a part of the theory of architecture and art as well. The main reason for this could be the 'self-propagating' nature of λ: If a point C_1 divides a segment $\overline{AB}$ so that $|AC_1| : |C_1B| = \lambda : 1$, and a point C_2 divides a segment $\overline{AC}$ so that $|AC_2| = |C_1B|$ then the point C_2 divides the segment $\overline{AC_1}$ again in the ratio $\lambda : 1$. This is equivalent to the condition $\frac{a}{x} = \frac{x}{a-x} = \lambda$, where $a = |AB|$ and $x = |AC_1|$.

Can you construct the golden section of a given segment AB with a ruler and compasses only?

I 30 ● **F 47** ↑ **I 27** ↑ **I 29** ↓ **I 31** ↓ **I 32** ↓ **I 62** ↓ **I 63**

Check whether the considerations in I 29 remain true for all $a > 0$ in the following identity:

$$\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}} = 1+\cfrac{a}{1+\cfrac{a}{1+\cfrac{a}{1+\cdots}}} = \frac{1+\sqrt{1+4a}}{2}.$$

The second identity together with the corresponding recurrence relation can be a basis to develop an algorithm of evaluation of $\sqrt{b}$ (at least for $b > 1$). Can you design such an algorithm and compare its effectiveness with that of I 27?

2

I 31 ●● **↓ PI 31** **↑ I 29 ↑ I 30 ↓ I 32**

Analyze the sequence

$$a_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots \sqrt{n}}}}.$$

Hint: Try monotonicity and boundedness of a_n.

I 32 ●●● **↓ PI 32** **↑ I 29 ↑ I 30 ↑ I 31**

Let $\{b_n\}$ be a given sequence of positive real numbers. Define the sequence x_n by

$$x_n = \sqrt{b_1 + \sqrt{b_2 + \sqrt{b_3 + \cdots \sqrt{b_n}}}}$$

and find conditions for its convergence.

Since 1202, when the Italian merchant and mathematician Leonardo of Pisa (ca. 1180–1250), better known as Fibonacci, introduced the following sequence, it found so many interesting applications in number theory, combinatorics, computer science and elsewhere, that it became probably the most studied infinite sequence ever. Originally Fibonacci posed the following problem: Suppose that some kind of rabbits live forever and that every month each pair bears a new pair which becomes productive from the second month on. If we start with one newborn pair, how many pairs of rabbits will there be in the n-th month?

I 33 •• ↓ **PI 33** ↑ **I 29**

Let F_n denote the number of pairs of rabbits in the n-th month. The sequence F_n is called the Fibonacci sequence. Find the recurrence formula for F_n.
From the vast number of interesting properties of this sequence, consider and prove the following:

(i) Two consecutive Fibonacci numbers are relatively prime.
(ii) Find $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ assuming that it exists. Compare the result with I 29.
(iii) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.
(iv) $F_{n+m} = F_m F_{n+1} + F_{m-1}F_n$. Deduce that F_{nk} is a multiple of F_k.
(v) Assume that $F_n = \lambda^n$ and find possible values of λ from the equation above.
(vi) Find an explicit expression for F_n with $F_1 = F_2 = 1$, using the values λ from (v).

Hint: In (ii) divide the recurrence formula by F_{n+1}. In (iii) and (iv) use induction. In (vi) put $F_n = a\lambda^n + b\lambda^{-1}$ for some a, b.

I 34 •• ↑ **I 01**

Take any fixed positive integer p and find, if it exists,

$$\lim_{n\to\infty} \left\lfloor \frac{n}{p} \right\rfloor \sin \frac{1}{\lceil \frac{n}{p} \rceil}.$$

Hint: Prove and take into account that

$$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1.$$

Start with small values of p. A plot of the functions Floor, Ceiling (in MATHEMATICA®) or floor, ceil (in Maple®) (with n replaced by x, $1 < x < 20$) may suggest the way of reasoning.

I 35 • ↑ **I 02** ↓ **I 36**

Let $a \neq 0$ and z_0 be complex numbers. Consider the sequence given by $z_{n+1} = az_n$. Find conditions under which $\lim_{n\to\infty} z_n$ exists.

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Hint: Try to find an explicit expression for z_n.

I 36 ●● **↑ I 02 ↑ I 35**

Let A be a real square matrix of order m and x_0 a vector of dimension m. Consider the sequence $x_{n+1} = Ax_n$. Find sufficient conditions for the existence of $\lim_{n\to\infty} x_n$.

Hint: You need some facts from linear algebra and matrix theory.

I 43 ●● **↑ I 01 ↑ I 02 ↑ I 34 ↑ I 35**

Let $A = \{a(1), a(2), \ldots, a(n)\}$ be a set of numbers arranged in the increasing way. Let $a \in A$ be given. To find its position in A we compare a with $a(\lfloor n/2 \rfloor)$ which divides A into its upper and lower part. We continue with one of these parts, depending where the number a fitted. Hence, the maximal number $B(n)$ of comparisons satisfies a certain recurrence formula.

(i) Can you find it?
(ii) Calculate some of first terms of $B(n)$ and prove that $B(n) = \lfloor \log_2 n \rfloor + 1$, where $\log_2$ denotes the logarithm to the base 2.
(iii) With the O-notation show that $B(n) = O(\log n)$.

Hint: Consider the binary representation of n and compare $\lfloor \log_2 n \rfloor$ with the number of digits in this representation.

The sequence $B(n)$ arises when estimating the computational complexity (e.g. the number of operations) of the so called 'divide and conquer algorithms' based on the following paradigm: Divide the problem into two subproblems of (approximately) equal size, solve them recursively, then use the solution to solve the original problem. An example of such problem is the binary search, where the position of an element in a sorted list is found by 'halving' the list successively. This paradigm is the basis of so-called 'fast' algorithms, e.g. the Fast Fourier Transform (FFT).

I 44 • ↑ I 35 ↑ I 36

In mathematical biology various models of the 'predator-prey interaction' are analyzed. A simple model is as follows: In an isolated area the number K of owls (in thousands) and the number L of mice (in millions) is considered stabilized. Due to some external factors this equilibrium has been destroyed. Denoting

$$x(n) = \text{the number of owls after } n \text{ years} - K,$$

$$y(n) = \text{the number of mice after } n \text{ years} - L,$$

the destruction of equilibrium can be characterized by setting $x(0)$ and $y(0)$ to be nonzero.
Recall that $\Delta x(n) = x(n+1) - x(n)$ and suppose that there exist positive constants $\alpha, \beta, \gamma, \delta$ all of them < 1, such that

$$\Delta x(n) = -\alpha x(n) + \beta y(n), \qquad \Delta y(n) = -\gamma x(n) - \delta y(n).$$

The first equation describes the fact that the decrease of $x(n)$ and increase of $y(n)$ might give more food to the remaining owls. The second equation reflects that the decrease of $x(n)$ means less danger for the mice and less mice gives the remaining mice less competition for food.
Prove that for $\alpha = \gamma = 0.1$, $\beta = 0.2$, $\delta = 0.4$ the initial deviation $x(0), y(0)$ will be made small after some period of time, i.e. that $\lim_{n\to\infty} x(n) = \lim_{n\to\infty} y(n) = 0$, which means that the original equilibrium will be restored. Can you find such values of the parameters that the original equilibrium will never be restored?

Hint: Write the equations in the form

$$\begin{pmatrix} x(n+1) \\ y(n+1) \end{pmatrix} = A \begin{pmatrix} x(n) \\ y(n) \end{pmatrix},$$

where $A = \left(\begin{smallmatrix} 1-\alpha & \beta \\ -\gamma & 1-\delta \end{smallmatrix}\right)$.

2

I 51 • ↑ **I 01** ↑ **I 03**

Find the cluster points of the following sequences:

$$a_n = \sin(n\pi/2), \qquad b_n = \sin(n^2\pi/3),$$

$$c_n = 3\cos(n\pi/2) + (-1)^n.$$

Hint: For b_n consider $n = 3k,\ 3k+1,\ 3k+2$.

I 52 ••• ↓ **PI 52** ↑ **I 03**

Prove that any point of the unit circle is a cluster point of the sequence $k_n = \exp(in)$, where $i^2 = -1$.

I 53 • ↑ **I 03** ↑ **I 52**

Prove that any point of the interval $[-1, 1]$ is a cluster point of the sequence $k_n = \sin n$.

I 54 • ↑ **I 53**

Given a finite set of reals $a_1, a_2, \dots, a_n$. Find a sequence x_n such that all cluster points of $\{x\}$ are exactly all the given a_i's. If the set consists of one single point a_1, is it possible to find a non-convergent sequence x such that a_1 is the only cluster point of x?

I 55 ••

Prove that for any sequence $\{a_n\}$ of positive real numbers with $\liminf a_n > 0$ there is

$$\frac{1}{\limsup a_n} = \liminf \frac{1}{a_n} \quad \text{and} \quad \frac{1}{\liminf a_n} = \limsup \frac{1}{a_n}.$$

Hint: Start with the case $0 < \liminf a_n \le \limsup a_n < \infty$ and show that for any set $M \subset (0, \infty)$, $\sup M = \inf\{\frac{1}{x} : x \in M\}$. Then use the definition of lim inf and lim sup.

I 61 • **↑ I 08 ↑ I 10 ↑ I 33 ↓ I 62 ↓ I 63 ↓ I 64 ↓ I 65**

Put $x_1 = a$, $x_2 = b$, where $a, b > 0$ and consider the sequences x_n defined by

(i) $x_n = (x_{n-1} + x_{n-2})/2$,
(ii) $x_n = \sqrt{x_{n-1}x_{n-2}}$,
(iii) $x_n = \dfrac{2}{\frac{1}{x_{n-1}} + \frac{1}{x_{n-2}}}$.

Find their limits if they exist.

Hint: In (ii) consider $\log x_n$, in (iii) consider $1/x_n$.

I 62 •• **↑ I 17 ↑ I 34 ↑ I 61**

Consider sequences $\{x_n\}$, $\{y_n\}$ defined by

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

with $x_0 = a$, $y_0 = b$, $a, b > 0$. Find their limits if they exist.

Hint: Show that $y_n \le x_n$ for $n \ge 1$ and that $x_{n+1} - y_{n+1} \le (x_n - y_n)/2$.

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I 63 ●●● **[M]** **M 30 M 33 E 01** ↑ **I 17** ↑ **I 62**

Consider sequences $\{x_n\}$, $\{y_n\}$ defined by

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_n y_n},$$

with $x_0 = a$, $y_0 = b$, $a, b > 0$. Prove the existence and equality of their limits. For a chosen a, b, evaluate the limit with an error $<10^{-4}$.

Hint: Show that for $n \geq 1$ there is $y_n \leq x_n$, x_n is decreasing and y_n is increasing. You may compare your result with results of the command `N[ArithmeticGeometricMean[a,b]]` or `GaussAGM` in MATHEMATICA® and Maple®, respectively.

This limit is called the arithmetic-geometric mean of a, b. Its value can be found using the Gauss theorem on elliptic functions (see Theorem H). It became an important tool in calculating the values of elliptic and some related special functions.

I 64 ●●● **[M]** ↑ **I 10** ↑ **I 62** ↑ **I 63**

Consider sequences $\{x_n\}$, $\{y_n\}$ defined by

$$x_{n+1} = \sqrt{x_n y_n}, \qquad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

with $x(0) = a$, $y(0) = b$, $a, b > 0$. Prove that both these limits exist and that they are equal. Find this limit in terms of the arithmetic-geometric mean and for given a, b, evaluate the limit with error $<10^{-4}$.

Hint: Consider $\xi_n = \frac{1}{y_n}$, $\eta_n = \frac{1}{x_n}$ and use I 63.

I 65 ••• [M] ↑ I 62 ↑ I 63 ↑ I 64

Consider sequences $\{x_n\}, \{y_n\}, \{z_n\}$ defined as follows

$$x_{n+1} = \frac{x_n + y_n + z_n}{3}, \qquad y_{n+1} = \sqrt[3]{x_n y_n, z_n}$$

$$z_{n+1} = \frac{3x_n y_n, z_n}{x_n y_n + x_n z_n + y_n z_n}$$

where x_0, y_0, z_0 are given positive numbers. Do the limits of these sequences exist? Are they mutually equal? If so, find this value (the AHG Mean of a, b, c) for given a, b, c with an error $<10^{-4}$.

Hint: Use the inequality for arithmetic, harmonic and geometric means (Theorem F) to show that $z_n \le y_n \le x_n$ for $n > 0$.

I 66 •• ↑ I 62 ↑ I 63 ↑ I 64

Denoting the limits of I 62, I 63, I 64 by AH, AG, HG, respectively the arithmetic-harmonic mean, arithmetic-geometric mean, harmonic-geometric mean, show that

$$HG \le AH \le AG$$

and equality holds true for $a = b$ only.

Hint: Use the inequality for arithmetic, harmonic and geometric means (Theorem F). To show e.g. that $AH \le AG$ prove that $x_n \le X_n$, $y_n \le Y_n$, for all $n \ge 0$, where x_n, y_n denote the sequences of I 62 and X_n, Y_n denote the sequences of I 63.

2

Supplementary Material

Definitions

A sequence $\{a_n\}$ of real or complex numbers is called convergent if there exists a number $A \in \mathbf{R}$ (or $\mathbf{C}$) such that for every $\varepsilon > 0$ there is an integer N such that $|a_n - A| < \varepsilon$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = A$. If the sequence is not convergent, we call it divergent.

A sequence $\{a_n\}$ of real numbers is called increasing if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$. Similarly, a sequence is called decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$.
A sequence $\{a_n\}$ is bounded (below, above, respectively) if there is a number L (m, M, respectively) such that $|a_n| \leq L$ ($a_n \geq m, a_n \leq M$, respectively) for all n.

Let $\{f_n\}$, $\{g_n\}$ be two sequences. We say that $f_n = O(g_n)$ for $n \to \infty$ if there exists a constant C (independent of n) such that $|f_n| \leq C|g_n|$ for all sufficiently large indexes n.
Let $\{a_n\}$ be a convergent sequence, $\lim_{n\to\infty} a_n = A$. We say that a sequence $\{a_n\}$ has the order of convergence of $\{g_n\}$ to A if $|a_n - A| = O(g_n), n \to \infty$.
If $\{a_n\}$ diverges to $\pm\infty$ we say that the sequence $\{a_n\}$ has the order of divergence of $\{g_n\}$ if $\frac{1}{a_n} = O(\frac{1}{g_n}), n \to \infty$.

For a sequence a_n we define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_m : m \geq n\}$$

and

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_m : m \geq n\}.$$

These numbers are called upper limit and lower limit, respectively.

For a sequence a_n of complex numbers the value A is called its cluster point if for any $\varepsilon > 0$ the inequality $|a_n - A| < \varepsilon$ is satisfied for an infinite number of indexes. (The smallest cluster point of a sequence of reals is its lower limit, the largest is its upper limit.)

Let $a = (a_1, a_2, \ldots, a_n)$ be an n-tuple of positive numbers. Their harmonic mean $H_n(a)$, geometric mean $G_n(a)$, and arithmetic mean $A_n(a)$ is defined are follows:

$$H_n(a) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}},$$

$$G_n(a) = \sqrt[n]{a_1 a_2 \ldots a_n}, \qquad A_n(a) = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

The integral $\int_0^{\pi/2}\sqrt{1-m\sin^2 x}\,dx = E(m)$ is called the complete elliptic integral of the second kind, $m \in [0, 1]$.
$\int_0^{\pi/2}\frac{1}{\sqrt{1-m\sin^2 x}}dx = K(m)$ is called the complete elliptic integral of the first kind.

Let A be a square $n \times n$ matrix. Any solution λ of the equation $\det(A-\lambda I)=0$ is called the eigenvalue of matrix A, where I denotes the unit matrix.

Theorems

A

Let $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$ be finite. Then

$$\lim_{n\to\infty}(a_n+b_n) = A+B, \qquad \lim_{n\to\infty}(a_n b_n) = AB.$$

If moreover $B \neq 0$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{A}{B}.$$

B

Every increasing bounded from above sequence (decreasing bounded from below sequence) has a finite limit.

C

(Bolzano–Cauchy theorem) A sequence $\{a_n\}$ is convergent iff for every $\varepsilon > 0$ there is an integer N such that $|a_n - a_m| < \varepsilon$ for any $n, m \geq N$.

D

(Squeeze theorem) Assume that $a_n \leq c_n \leq b_n$ for all n sufficiently large. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = A$, then also $\lim_{n\to\infty} c_n = A$.

E

(Fixed point theorem) Let f be a real-valued function defined on the interval (a, b) such that its range belongs to (a, b). If there exists a number α, $0 \leq \alpha < 1$, such that for all $(x_1, x_2) \in (a, b)$ there is

$$|f(x_1) - f(x_2)| \leq \alpha|x_1 - x_2|,$$

then there exists exactly one value x such that $f(x) = x$.

2

F

For any n-tuple of positive numbers $a = (a_1, a_2, \dots, a_n)$ there is

$$\min(a) \le H_n(a) \le G_n(a) \le A_n(a) \le \max(a)$$

with equality iff $a_1 = a_2 = \cdots = a_n$ (see I 19).

G

Let A be a square $m \times m$ matrix. If the moduli of all eigenvalues of A are less than 1, then $\lim_{n\to\infty} A^n = 0$, where 0 is the zero matrix.

H

Let $a, b > 0$, put $q = \max(a, b),\ p = \min(a, b)$. Then

$$\int_0^{\pi/2} \frac{1}{\sqrt{a^2\cos^2 x + b^2 \sin^2 x}} dx = \frac{1}{q} K\left(1 - \frac{p^2}{q^2}\right).$$

Moreover, Gauss proved that

$$\int_0^{\pi/2} \frac{1}{\sqrt{a^2\cos^2 x + b^2 \sin^2 x}} dx = \int_0^{\pi/2} \frac{1}{\sqrt{r^2\cos^2 x + s^2 \sin^2 x}} dx$$

when $r = \frac{a+b}{2}$ and $s = \sqrt{ab}$.

Plans of Solution

PI 08

1. It is sufficient to consider $A = 0$.
2. For any $n, p \in \mathbf{N}$ there is

$$|y_{n+p}| \le \left| \frac{x_1 + x_2 + \cdots + x_n}{n+p} \right| + \frac{p}{n+p} \max(|x_{n+1}|, |x_{n+2}|, \dots, |x_{n+p}|).$$

3. Deduce that $\lim_{n\to\infty} y_n = 0$.

PI 22
1. Show that both $x(n)$ and $y(n)$ are decreasing and bounded. Find that $x(n) = y(n)$ for all $n \geq 0$.
2. Using integration by parts show that

$$x(n-1) \geq x(n) = \frac{n-1}{n} x(n-2) \geq \frac{n-1}{n} x(n-1).$$

3. Deduce that $\lim_{n\to\infty} \frac{x(2n+1)}{x(2n)} = 1$.
4. Show that

$$\frac{x(2n+1)}{x(2n)} = \frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{2}{\pi}$$

and thus deduce that

$$\lim_{n\to\infty} \frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 = \frac{\pi}{2}.$$

PI 23
1.Show that the sequence y_n is dominated by the convergent sequence $\pi^{1/n}$.
2. For $0 < \varepsilon < \frac{\pi}{2}$ put $A_\varepsilon = [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. Show that

$$y(n) \geq \left(\int_{A_\varepsilon} \sin^n x dx\right)^{\frac{1}{n}}.$$

PI 24
1. The function f attains its maximum at some point $x_0 \in [a, b]$.
2. Show that $x_n \leq |f(x_0)|(b-a)^{1/n}$.
3. Show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in [x_0 - \delta, x_0 + \delta]$ we have

$$|f(x)| \geq |f(x_0)| - \varepsilon.$$

4. Prove that $x_n \geq (|f(x_0)| - \varepsilon)(2\delta)^{\frac{1}{n}}$.

PI 25
1. Show that x_n is increasing: use the Bernoulli inequality

$$(1+x)^n \geq 1 + nx, \quad x > -1,\ n \in \mathbf{N}$$

in proving that $\frac{x_{n+1}}{x_n} \geq 1$.
2. Show that $y_n = (1 + \frac{1}{n})^{n+1}$ is decreasing by proving that $\frac{y_n}{y_{n+1}} \geq 1$ and that $x_n \leq y_n$.
3. Show that $\lim_{n\to\infty} x_n$ exists. It is commonly denoted by e and it is the base of natural logarithms.

2

PI 26

Decipher the following command written in MATHEMATICA® language.

```
BrokenLine[f_.a_,x0_,b_,n_]:=Module[{},
li=NestList[f,x0,n];
br=Line[Flatten[Table[li[[k]],li[[k+1]],
li[[k+1]],li[[k+1]],k,1,n-1],1]];
Show[Plot[x,f[x],x,a,b],Graphics[br]]]
```

For Maple® see the Maple® worksheet.

PI 29

1. Denote $u_n = \sqrt{1+\sqrt{1+\cdots\sqrt{1}}}$, where the square root is repeated n times, $u_0 = 0$. Show that u_n is increasing and bounded (by induction). Find the relation between u_{n+1} and u_n, and then $\lim_{n\to\infty} u_n$.
2. Define v_n by

$$v_n = 1 + \frac{1}{1+\cdots+\frac{1}{1}}, \qquad v_0 = 1$$

(with n appearances of the fraction line). Plot the first few values of v_n. Find the recurrence formula for v_n.
3. Use induction to show that $1 \le v_n \le 2$ and decide whether v_n is monotonic.
4. Show that the subsequence v_{2n} is increasing and the subsequence v_{2n-1} is decreasing. Find their limits.
5. Alternatively, show that

$$\left| v_n - \frac{1+\sqrt{5}}{2} \right| \le \left(\frac{2}{1+\sqrt{5}} \right)^n \left| v_0 - \frac{1+\sqrt{5}}{2} \right|.$$

PI 31

1. Show that a_n is increasing.
2. Use n-times the identity $a - b = (a^2 - b^2)/(a+b)$ to show that

$$a_{n+1} - a_n \le \frac{\sqrt{n+1}}{2^n \sqrt{n!}} = \frac{n+1}{2^n \sqrt{(n+1)!}}.$$

3. Show that

$$a_n \le \sum_{k=0}^{\infty} \frac{k+1}{2^k \sqrt{k!}}.$$

4. Deduce that $\lim_{n\to\infty} a_n$ exists. Numerical experiments may show how this limit differs from the sum above.

PI 32

1. The sequence x_n is increasing. Hence its limit, possibly infinite, exists.
2. Use the fact that if there exists a subsequence n_k of indexes such that

$$\frac{\log b_{n_k}}{2^{n_k}} \to \infty$$

then $\lim_{n\to\infty} x_n = \infty$ to show that $x_n \geq (b_n)^{2^{-n}}$.
3. Assume that there exists a $q > 0$ such that $b_n \leq q^{2^n}$.
Use the result of I 29 to show that $x_n \leq q(1+\sqrt{5})/2$.
4. Conclude that a finite limit of $\{x_n\}$ exists if and only if

$$\limsup_{n\to\infty} \frac{\log b_n}{2^n} < \infty.$$

PI 33

In Fibonacci's problem setting the number of pairs in the first and second month is 1. Hence $F_1 = F_2 = 1$. In the $(n+2)$-th month the number of pairs equals to the sum of the number of pairs in the previous month and the number of pairs of newborn rabbits, i.e. $F_{n+2} = F_{n+1} + F_n$.

(i) If $\gcd(F_{m+1}, F_m) = s > 1$ then all F_n should be divisible by s, which is evidently wrong.

PI 52

1. For any given integer m divide the interval $(0, 2\pi)$ into m subintervals of equal length $2\pi/m$.
2. Considering numbers $k \mod 2\pi$, $k = 1, 2, \ldots, m+1$, show that two of them belong to the same subinterval of length $2\pi/m$.
3. Since π is not rational, all integer multiples of the difference of these two numbers (still $\mod 2\pi$) form a set, which has the following property: The distance from any point of the interval $(0, 2\pi)$ to some point of this set is at most $1/m$.
4. Since m is an arbitrary integer we obtain that any point of the interval $(0, 2\pi)$ is a cluster point of the sequence $a_n = n \mod 2\pi$.

Further References

Infinite sequences are dealt with in any standard textbook of Calculus.

Elliptic functions are treated in a number of monographs, e.g.

Whittaker E.T, Watson G.N., A Course of Modern Analysis, Cambridge University Press, 1927,

Hurwitz A., Courant R., Vorlesungen uber Allgemeine Funktionentheorie und Elliptische Funktionen, Springer, Berlin, 1964, and many others.

Implicitly given sequences are often solutions of difference equations. Again, a number of monographs are devoted to this topic. One of the earliest and one of the latest, where additional references can be found, are the books:

Poincaré H., Sur les equation lineaires aux differentielles ordinaires at aux differences finies, Am. J. Math. 7 (1885), 203–258.

Kelley W.G., Peterson A.C., Difference Equations, An Introduction with Applications, Academic Press, New York, 2001.

Periodicity 3

Perception of events occurring at regular time or space intervals may be one of the basic instincts of animals. The sense of rhythm proving itself in dance is almost as old as mankind. Observation of regularly occurring events in the sky led to basic discoveries in astronomy. Periodicity is the abstract concept of such regularities. Its importance lies in predictability of events; violation of periodicity signals malfunction of the system or external perturbancies. Mathematics, wanting to describe and model processes in nature, life and man-made objects must devote considerable attention to the concept of periodicity. Due to the importance of periodic motion in science and technology we will include some problems concerning periodical solutions of differential and difference equations. One of the most important tools in investigating periodicity is the theory of Fourier series. Only its rather formal description is included here without dealing with more advanced topics of the theory of convergence of Fourier series.

> In this chapter we want to deal with various aspects of the periodicity concept as it appears in different parts of analysis. Knowledge of some basic facts on differential and difference equations and Fourier series is assumed.
> The solution of most of the problems can be made easier when using the computational power of MATHEMATICA®.
> The commands
> `Plot, ListPlot, ListPlot3D, Nest, Inverse,`
> `Mod, Manipulate, DSolve, Integrate, NDSolve, RSolve,`
> `Eliminate`
> and related ones can help.
> Similarly in Maple® the commands `plot, int, mod,` and commands contained in the
> `Interactive Plot Builder Assistant, Linear Algebra`
> `package, ODE Analyzer Assistant, plots[animate]`
> may help.
> The reader is encouraged to learn the syntax and semantics of the commands and examples of their use. This information is contained on the corresponding help-page.

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In most of the examples numerical experiments may give a starting point of reasoning or verify initial conjectures. Explanation of mathematical terms and concepts can be found on the Internet, e.g. at www.mathworld.wolfram.com or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://eom.springer.de.

Suggestions

- Simple examples to start with are P 01, P 02, P 03, P 06, P 12, P 13.
- Periodicity in the behavior of various dynamical systems is the theme of P 14 – P 24. Compare results obtained for linear differential and difference equations. It is also recommended to formulate and solve actual examples for each of the problems. P 20 gives a simple outlook to the vast area of periodical motions of nonlinear systems.
- Some basic examples of Fourier series are given in P 09, P 10, P 11.
- More involved cases of the periodicity concept are dealt with in P 08, P 25, P 26, P 27 and periodicity of two variable sequences is a topic in P 28 – P 36.

Problems

P 01 • ↓ **P 03** ↓ **P 04**

Find a period of the following functions, if it exists.

(a) $f(x) = \sin(\omega x + \alpha)$, $\alpha, \omega \in \mathbf{R}$;
(b) $f(x) = \cosh \lambda x$, $\lambda \in \mathbf{R}$;
(c) $f(n) = n(\operatorname{mod} m)$, $n \in \mathbf{Z}$, $m \in \mathbf{N}$ fixed;
(d) $f(x) = \lambda x - [\lambda x]$, $\lambda \in \mathbf{R}$, $\lambda \neq 0$, where $[x]$ is the integer part of x;
(e) $g(x) = \lambda x - \lambda[x]$, $\lambda \in \mathbf{R}$, $\lambda \neq 0$,
(f) Prove that a function f satisfying $f(x) = -f(x+T)$ for all $x \in \mathbf{R}$ and for a fixed value $T > 0$ is periodic with period $2T$.

P 02 • ↓ P 03 ↓ P 04

Let f be the so-called Dirichlet function defined by

$$f(x) = \begin{cases} 1 & \text{for } x \text{ rational}, \\ 0 & \text{for } x \text{ irrational}. \end{cases}$$

Show that every positive rational number is a period of f, hence f has no primitive period. Is also any irrational number a period of f?

P 03 • ↑ P 01 ↓ P 05 ↓ P 12 ↓ P 13

Let f, g be periodic functions of equal periods. Examine for periodicity the following functions:

$$\alpha f + \beta, \quad \alpha f + \beta g, \quad fg, \quad f/g, \quad f'$$

provided all these functions are well defined on **R**.

P 04 ••

Let f be a periodic function of period T. Prove that the following identity holds for any real number a:

$$\int_0^T f(x)dx = \int_a^{a+T} f(x)dx.$$

P 05 • ↑ P 04 ↓ P 12 ↓ P 13

Let f, g be periodic functions of period T which are locally integrable on **R**. Examine for periodicity the functions:

$$\int_0^x f(t)dt, \qquad \int_0^x f(t)g(t)dt.$$

3

P 06 • ↑ **P 01** ↑ **P 03** ↓ **P 07**

Find the primitive period of the function

$$h(x) = 2\sin(x/2) + 5\sin 2x - \cos(x/5),$$

if it exists.

P 07 • ↓ **P 08** ↓ **P 13**

Show that the sum of two periodic functions with periods T_1 and T_2 is again periodic if T_1/T_2 is rational.

P 08 • • • ↓ **PP 08** ↑ **P 07** ↓ **P 31**

Assume that f_1 and f_2 are non-constant continuous periodic functions with periods T_1, T_2, respectively. Prove that if T_1/T_2 is irrational then the function $f_1 + f_2$ is not periodic.

P 09 •• **[M]**

Find the Fourier series of the function
$f(x) = \operatorname{sign} x$, $-1 < x < 1$ with period $T = 2$. Plot the graphs of a few partial sums $s_n(x)$. Observe the behavior of these partial sums in a neighborhood of $x = 0$.
Give your estimate of the maximal difference between the value of the function $f(x)$ and the value of the partial sums, i.e. $\max_{|x|<\delta} |f(x) - s_n(x)|$ for various values of n.
For comparison, repeat these tasks for the function $g(x) = 1 + \operatorname{sign}(x - 1/2)$, $0 < x < 1$ with period $T = 1$ and for other non-continuous functions of your choice.

The partial sums of a Fourier series of a periodic function f approximate the values of f in a specific way. Investigations of this approximation form the basic content of the theory of Fourier series. During the first decades of the 20-th century some rather controversial results in this direction led to a thorough revision of basic concepts of mathematical analysis, among others a basic revision of the theory of integrals obtained by H. Lebesgue. This example points towards the so-called Gibbs phenomena.

P 10 ●● **F 34** ↑ **P 09**

Denote the n-th partial sum of a Fourier series of a general 2π-periodic function f by s_n and prove that

$$s_n(x) = \int_0^{2\pi} D_n(x-t)f(t)dt,$$

$$\text{where } D_n(u) = \frac{1}{2\pi}\frac{\sin(n+1/2)u}{\sin u/2}.$$

Hint: In the partial sum of the Fourier series at a fixed point x_0 replace the coefficients by their integral representation and use the result of F 34.

The function $D_n(u)$ is called the Dirichlet kernel. It is of basic importance when investigating convergence of Fourier series.

P 11 ●● ↑ **P 01** ↑ **P 09**

Consider 2π-periodic functions f satisfying one of the following conditions:

(a) $f(t) = f(-t)$,
(b) $f(t) = -f(-t)$,
(c) $f(t) = f(-t)$ and for $t \in (0,\pi) : f(t) = -f(t+\pi)$,
(d) $f(t) = f(-t)$ and for $t \in (0,\pi) : f(t) = f(t+\pi)$,
(e) $f(t) = -f(-t)$ and for $t \in (0,\pi) : f(t) = -f(t+\pi)$,
(f) $f(t) = -f(-t)$ and for $t \in (0,\pi) : f(t) = f(t+\pi)$.

Their graphs exhibit some symmetries. Describe these symmetries. In each case infinitely many of their Fourier coefficients equal zero and the corresponding Fourier series has one of the following forms:

(i) $\sum_{k=1}^{\infty} a_{2k+1}\cos((2k+1)t)$,

(ii) $\sum_{k=1}^{\infty} b_{2k+1}\sin((2k+1)t)$,

(iii) $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_{2k}\cos(2kt)$,

(iv) $\sum_{k=1}^{\infty} b_{2k}\sin(2kt)$,

(v) $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kt)$,

(vi) $\sum_{k=1}^{\infty} b_k\sin kt$.

Can you assign some cases under (a), (b), (c), (d), (e), (f) to those under (i), (ii), (iii), (iv), (v), (vi) ?

P 12 •• ↑ P 03 ↑ P 05 ↑ P 07 ↓ P 25

Let f, g be periodic sequences of equal period T. Examine for periodicity the following sequences:

$$\alpha f + \beta, \quad \alpha f + \beta g, \quad fg, \quad f/g$$

and also the sequences d and a defined by

$$d(n) = f(n+1) - f(n), \qquad a(n) = \sum_{k=1}^{n} f(k).$$

P 13 •• ↑ P 03 ↑ P 05 ↑ P 07 ↓ P 25

Consider the sequences f and g of unequal periods T_1 and T_2, respectively. What can be said about the periodicity of

$$\alpha f + \beta, \quad \alpha f + \beta g, \quad fg, \quad f/g?$$

Compare the answers with those of P 03, P 05.

P 14 • ↑ P 03

Consider the linear differential equation $\dot{x} + ax = 0$, where $a \in \mathbf{C}$. For which values of a are all its solutions periodic?

P 15 •• ↑ P 03

Find conditions for the coefficients of a second order homogeneous linear differential equation $\ddot{x} + a\dot{x} + bx = 0$ with constant coefficients $a, b \in \mathbf{C}$ to have only periodic solutions.

P 16 ●● ↑ **P 03** ↑ **P 15**

Find conditions for the coefficients of a homogeneous linear differential equation of order n with constant complex coefficients to have only periodic solutions. How would your answer change if only real coefficients are allowed?

P 17 ●● ↑ **P 03** ↑ **P 15** ↓ **P 24**

Let the right-hand side of an n-th order linear differential equation with constant coefficients be a trigonometric polynomial of degree n with period $T > 0$. Find necessary and sufficient conditions for the coefficients of the equation such that the equation has a periodic solution.

Hint: Start with a (complex) polynomial of degree 1.

P 18 ● ↑ **P 14** ↑ **P 15** ↓ **P 19** ↓ **P 20**

Consider the following system of differential equations:

$$x' = a_{11}x + a_{12}y, \qquad y' = a_{21}x + a_{22}y$$

for $x = x(t)$, $y = y(t)$. Find conditions for the coefficients a_{ik} such that its solution is periodic for all initial conditions.

Hint: Differentiate the first equation with respect to t and exclude y.

P 19 ● ↑ **P 14** ↑ **P 15** ↑ **P 18** ↓ **P 20**

Find necessary and sufficient conditions for the complex coefficient α such that any complex solution z of the differential equation $z' = \alpha z$, $z \in \mathbf{C}$ is a (complex) periodic function.

Hint: Consider the real and imaginary parts of the equation.

3

P 20 ••• [M] ↑ **P 18** ↑ **P 19**

Examine the behavior of solutions of the (complex) differential equation $z'(t) = e^{\mathbf{i}z(t)} - 1$ with real variable t and with an initial condition $z(0)$ such that $|z(0)|$ is small. Show that some of these initial conditions lead to a periodic solution.

Hint: Rewrite the equation as a system of two real equations and use software tools applying numerical solutions of ordinary differential equations.

P 21 • ↑ **P 03** ↑ **P 12** ↑ **P 13** ↑ **P 14**

Consider the linear difference equation $x(n+1)+ax(n)=0$ with $a \in \mathbf{C}$. For which values of a are all its solutions periodic?

P 22 •• ↑ **P 03** ↑ **P 12** ↑ **P 13** ↑ **P 15**

Find conditions for the coefficients of a second order homogeneous linear difference equation $x(n+2) + ax(n+1) + bx(n) = 0$ with constant coefficients $a, b \in \mathbf{C}$ to have only periodic solutions.

P 23 •• ↑ **P 03** ↑ **P 12** ↑ **P 13** ↑ **P 16**

Find conditions for the coefficients of a homogeneous linear difference equation of order n with constant complex coefficients to have only periodic solutions. How would your answer change if only real coefficients are allowed?

P 24 •• ↑ **P 03** ↑ **P 12** ↑ **P 13** ↑ **P 17**

Let the right-hand side of an n-th order linear difference equation with constant coefficients be a periodic sequence of period T. Find necessary and sufficient conditions for the complex coefficients of the equation such that the solution of the equation with zero initial conditions is a periodic function.

P 25 ••• ↓ PP 25 ↑ P 06 ↑ P 08

Prove that there exist periodic functions f_1 and f_2 on $\mathbf{R}$ with periods T_1, T_2 respectively, such that T_1/T_2 is irrational and $f_1 + f_2$ is periodic with period $T_1 + T_2$. (Such functions cannot be continuous!)

P 26 •• M 50

(a) Find all mappings of the form $g(x) = \frac{ax+b}{cx+d}$ satisfying $g(g(x)) = x$.
(b) Find all mappings of the form $f(x) = \frac{ax+b}{cx+d}$ satisfying $f(f(f(x))) = x$.

P 27 ••• M 50 ↓ PP 27 ↑ P 23

Find all mappings of the form $f(x) = \frac{ax+b}{cx+d}$ which are k-periodic with respect to superposition, i.e. k is the smallest integer satisfying $f^{[n+k]} = f^{[n]}$ for all n, where $f^{[n]}$ denotes the n-fold superposition ($f^{[n+1]} = f(f^{[n]})$, $f^{[0]}(x) = x$).

P 28 •• ↓ P 29

Consider the function $f(m,n) = \sin\frac{2\pi(m+n)}{3}\cos\frac{2\pi(m-2n)}{3}$ on $\mathbf{Z}^2$.

(a) Show that $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ are both periodicity matrices of f.
(b) Can you find other matrices of periodicity which are not integer multiples of either A or B?
(c) How can all periodicity matrices of f be created?

P 29 • ↑ P 28

Consider the function $g(m,n) = \sin\frac{(3m+2n)\pi}{5}$. Show that it is a periodic function with a singular periodicity matrix. Find this matrix.

3

P 30 •

↑ P 03 ↑ P 12

Let functions $f : \mathbf{Z}^2 \to R$, $g : \mathbf{Z}^2 \to R$ be periodic with a periodicity matrix A. Examine for periodicity the following functions:

$$\alpha f + \beta, \quad \alpha f + \beta g, \quad fg, \quad f/g,$$

with $\alpha, \beta \in \mathbf{R}$.

P 31 ••

↑ P 03 ↑ P 12 ↑ P 13

Let functions $f : \mathbf{Z}^2 \to R$, $g : \mathbf{Z}^2 \to R$ be periodic with periodicity matrices A, B, $A \neq B$, respectively. Find conditions under which the functions $\alpha f + \beta g$, fg, f/g are periodic.

P 32 ••

F 42 F 44

All values of a function $f : \mathbf{Z}^2 \to \mathbf{R}$ periodic with a nonsingular periodicity matrix A are determined by a finite number of its values. Find this number in terms of the matrix A and give a geometric interpretation of their location in $\mathbf{Z}^2$.

P 34 ••

↑ P 32

Let $A = (a_{ik})$ be a periodicity matrix of $f : \mathbf{Z}^2 \to \mathbf{R}$. Then the (integer) point (x, y) lies inside the 'basic parallelogram' iff the system of linear equations with an 'augmented' matrix A^+ with the three rows

$$(a_{11}, a_{11} + a_{12}, a_{12}), (a_{21}, a_{21} + a_{22}, a_{22}), (1, 1, 1)$$

and with the right-hand side $\{x, y, 1\}^T$ has a positive solution. Prove this statement. Can you find an equivalent and simpler characterization of the points belonging to this parallelogram?

P 35 •• ↑ P 32 ↑ P 34

Let $f : \mathbf{Z}^2 \to \mathbf{R}$ be a periodic function with the periodicity matrix A and with given values in all points of the 'basic parallelogram'. Find a procedure to evaluate its value at an arbitrary integer point.

P 36 ••• ↓ PP 36 ↑ P 34

Consider a hexagonal tiling of the plane with the regular hexagon H circumscribed to the unit circle as its basis and with two of its sides parallel to the x axis. Let F be a function defined at the points of H. Find its value at an arbitrary point $(x, y) \in \mathbf{R}^2$, given that $F(x, y) = F(x_0, y_0)$, where $(x_0, y_0) \in H$ and (x, y) are congruent with respect to the tiling. Is this function periodic? If so, can you find its periodicity matrix?

Supplementary Material

Definitions

A function $f : \mathbf{R} \to \mathbf{R}^n$ is called periodic if there exists a positive number $T \in \mathbf{R}$ such that

$$f(x) = f(x + T) \text{ for all } x \in \mathbf{R}.$$

The smallest of such T if it exists is called the primitive period of f.
Remark: This definition is also used with $\mathbf{R}$ replaced everywhere by $\mathbf{R}_+$, $\mathbf{N}$ or $\mathbf{Z}$.

A function $f : \mathbf{R}^2 \to \mathbf{R}$ is called periodic with a matrix of periodicity $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ iff

$$f(x, y) = f(x + ma_{11} + na_{12}, y + ma_{21} + na_{22})$$

for all $x, y \in \mathbf{R}$ and for all integer values m, n.
The parallelogram with vertices $(0, 0)$, (a_{11}, a_{21}), $(a_{11} + a_{12}, a_{21} + a_{22})$, (a_{12}, a_{22}) is called its basic parallelogram.
Remark: This definition is also used with $\mathbf{R}^2$ replaced by $\mathbf{N}^2$ or $\mathbf{Z}^2$ and with $a_{ik} \in \mathbf{N}$ or $\mathbf{Z}$, respectively.

3

For $T > 0$ and a real function f with $\int_0^T |f(t)|dt < \infty$ the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\omega x + b_k \sin k\omega x)$$

with $\omega = \frac{2\pi}{T}$ and with

$$a_k = \frac{1}{T}\int_0^T f(t)\cos k\omega t dt, \qquad b_k = \frac{1}{T}\int_0^T f(t)\sin k\omega t dt$$

is called the Fourier series of the function f.

The function $g : \mathbf{R} \to \mathbf{C}$ with

$$g(x) = c_0 + \sum_{k=-n}^{n} c_k \exp(\mathbf{i}k\omega x)$$

where $\omega = \frac{2\pi}{T}$, $T > 0$, is called a (complex) trigonometric polynomial of degree n.
The function $h : \mathbf{R} \to \mathbf{R}$ with

$$h(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos k\omega x + b_k \sin k\omega x)$$

where $\omega = \frac{2\pi}{T}$, $T > 0$ is called a (real) trigonometric polynomial of degree n. Such a trigonometric polynomial is a periodic function of period T.

Theorems

A

A tiling (also called a tessellation) of the plane with regular n-gons exists iff $n = 3, 4, 6$.

Plans of Solution

PP 08

1. For every $\epsilon > 0$ there exist integers p, q such that $|pT_1 - qT_2| < \epsilon$. Indeed, consider a partition of the interval $[0, T_1]$ into subintervals of equal length $d < \epsilon$. For $n \geq 1 + T_1/d$ form the set $\{\mathrm{mod}[kT_2, T_1],\ k = 1, 2, \dots, n\}$. Then at least two of its elements, say the q_1-th and q_2-th, belong to one of the intervals, i.e. $|\,\mathrm{mod}[q_1 T_2, T_1] - \mathrm{mod}[q_2 T_2, T_1]| < \epsilon$.

   Hence there are $p_1, p_2 \in \mathbf{Z}$ such that $|(q_1 T_2 + p_1 T_1) - (q_2 T_2 + p_2 T_1)| < \epsilon$. Put $p = p_1 - p_2$ and $q = q_2 - q_1$.
2. The set $\mathrm{mod}[kT_2, T_1], k \in \mathbf{Z}$ is dense in $[0, T_1]$.

   Let $\epsilon > 0$ be arbitrarily small. Then there exist p, q such that $|pT_1 - qT_2| = \epsilon_0 < \epsilon$. Hence either $qT_2 - pT_1 = \epsilon_0$ and so

   $$\mathrm{mod}[qT_2, T_1] = \epsilon_0, \qquad \mathrm{mod}[2qT_2, T_1] = 2\epsilon_0, \qquad \dots$$

   or $pT_1 - qT_2 = \epsilon_0$ and so

   $$\mathrm{mod}[qT_2, T_1] = T_1 - \epsilon_0, \qquad \mathrm{mod}[2qT_2, T_1] = T_1 - 2\epsilon_0, \qquad \dots .$$

   In both cases the set $\mathrm{mod}[kqT_2, T_1],\ k \in \mathbf{Z}$ has the property that any (fixed) point in $[0, T_1]$ is at a distance at most ϵ from some point of $\mathrm{mod}[kqT_2, T_1], k \in \mathbf{Z}$.
3. Assume that $T_1 < T_2$ and f_i attain their maximum at $x_i \in [0, T_1]$, $i = 1, 2$. By Step 2 we conclude that the set $\mathrm{mod}[x_1 + k_2 T_2, T_1]$ is dense in $[0, T_1]$. Hence there are $p_1, p_2 \in \mathbf{Z}$ such that

   $$|x_1 + p_1 T_1 - (x_2 + p_2 T_2)| < \epsilon. \quad (*)$$

4. Show that (*) and the continuity of f_1 and f_2 implies that

   $$\max(f_1 + f_2) = \max f_1 + \max f_2.$$

5. Assume that $f_1 + f_2$ has period T. Then either T/T_1 or T/T_2 is irrational.
6. Assume that T/T_1 is irrational and show using Step 4 that the maximal value of f_1 is repeated with periods T and T_1. Since the set of such points is dense in $[0, T_1]$, f_1 is constant.

3

PP 25

1. Consider R as a linear space over the field Q and let $(x_\alpha)_\alpha$ denote its basis (called the Hammel basis with the following meaning: every $x \in R$ can be written as a finite combination of some elements of this basis with rational coefficients).
2. Show that the ratio of any two non-identical elements x_α, x_β is irrational.
3. Chose $x_\alpha \neq x_\beta, x_\alpha, x_\beta > 0$ and denote $T_1 = x_\alpha T_2 = x_\beta$ and define

$$f_1(x_y) = \begin{cases} 1 & \text{if } \gamma \neq \alpha, \beta, \\ 0 & \text{if } \gamma = \alpha, \\ 1/2 & \text{if } \gamma = \beta, \end{cases}$$

$$f_2(x_y) = \begin{cases} 1 & \text{if } \gamma \neq \alpha, \beta, \\ 0 & \text{if } \gamma = \alpha, \\ -1/2 & \text{if } \gamma = \beta. \end{cases}$$

4. Extend linearly f_1 and f_2 to R and show that f_1 has period T_1 and f_2 has period T_2.
5. Show that $f_1 + f_2$ has period $T_1 + T_2$.

PP 27

1. Represent the function $f(x) = \frac{ax+b}{cx+d}$ by the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and show that the superposition $f^{[n]}$ can be represented by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^n$.

   Without loss of generality it can be assumed that $\det A = 1$.
2. Show that for any $t \in \mathbf{R}$ there is

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}^k = \begin{pmatrix} \cos kt & \sin kt \\ -\sin kt & \cos kt \end{pmatrix}.$$

3. For a given k define

$$f(x) = \frac{x \cos \frac{2\pi}{k} + \sin \frac{2\pi}{k}}{-x \sin \frac{2\pi}{k} + \cos \frac{2\pi}{k}}$$

   and show that this function satisfies the periodicity requirements.

PP 36

1. In a hexagonal tiling chose the coordinate system such that the origin coincides with one of the centers of hexagons, the x-axis is parallel to two of its sides and the radius of the inscribed circle equals one. Denote the points inside this hexagon or on three of its neighboring sides (x_0, y_0) and call this hexagon the basic one.
2. Show that the centers of hexagons have coordinates $(m\sqrt{3}, n)$ such that $m, n \in \mathbf{Z}$ and m, n are of the same parity.
3. Show that the point (x, y) belongs to the basic hexagon iff

$$-1 \leq y < 1 \quad \text{and} \quad -2 \leq y \pm x\sqrt{3} < 2.$$

4. Any point (x, y) can be expressed as $(x_0 + p\sqrt{3}, y_0 + q)$ with integers p, q of equal parity. To find x_0, y_0 for a given (x, y) solve the inequalities in Step 3 to find the pair p, q.
5. To find a periodicity matrix consider any two vectors with endpoints in the centers of two hexagons.

Further References

Kammler D.W., A First Course in Fourier Analysis, Prentice Hall, Upper Saddle River, 2000.

Part II
Tools

Finite Sums 4

Finite sums pose a problem if the number of summands is large and/or when the evaluation of each of the summands has a common and non-simple pattern. Simplification of such sums demands special methods and skill. The use of computers opened new problems in this directions.

In this chapter you will learn some techniques of dealing with finite sums. You will find out that computers cannot only calculate but they can also be used for proving some results, hence in certain cases computers may be considered as 'proving machines'. The solution of problems marked as [M] and many others can be made easier when using MATHEMATICA®, although only very few of them can be solved directly. The reader is strongly advised to learn the syntax and semantics of some of the commands. In MATHEMATICA® the following commands (and related ones) may help:

```
{\tt Sum, Series, BaseForm, IntegerDigits,
DigitQ, Fibonacci, HarmonicNumber,\dots}
```

In Maple®:

```
sum, series, convert, StringTools[IsDigit],
comninat[fibonacci], harmonic
```

In most of the examples numerical experiments may give a starting point for reasoning or help to verify initial conjectures. Explanation of mathematical terms and concepts can also be found on the Internet, e.g. at www.mathworld.wolfram.com or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://eom.springer.de.

J. Gregor, J. Tišer, *Discovering Mathematics*,
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4

Suggestions

- A good start for beginners are examples F 01, F 03, F 30 and further along the downward arrows.
- Those who are research inclined will find interesting stimuli in F 24, F 25, F 38, F 45, F 48 and may use the upward arrows to find help.
- Teachers could use F 09, F 15, F 52, F 54 to motivate subsequent work.
- All readers are strongly encouraged to modify, generalize or simplify the formulated problems, to find alternative formulations and formulate and solve their own examples and compare the context of these problems to the given ones.

Problems

Finite sums can be written explicitly as $a_1 + a_2 + a_3 + a_4 + a_5$ or with some obvious terms deleted, e.g. as $a_1 + a_2 + a_3 + a_4 + \cdots + a_{20}$. This may sometimes be ambiguous. More common is the so-called Sigma–notation: $\sum_{k=0}^{n} a_k$. In this sum exactly those a_k are included whose index k is an integer lying at or between the lower limit 0 and the upper limit n. The lower and upper limits can be replaced by other conditions which determine the terms included:

$$\sum_{0\le k^3\le 20} k = 0+1+2=3.$$

When no such terms exist, an 'empty' sum is by convention set equal to zero. The double sum $\sum_{i,k=1}^{n} a_{ik}$ is understood as $\sum_{i=1}^{n}(\sum_{k=1}^{n} a_{ik}) = \sum_{k=1}^{n}(\sum_{i=1}^{n} a_{ik})$.

A change of the order of summation or change of the variables may help to evaluate such multiple sums.

F 00 • ↓ **F 11**

Evaluate the following sums:

$$\sum_{0\le k^2\le 20} k(-1)^{k(k-1)/2}, \qquad \sum_{k^2+k+1\le 0} k(k+1).$$

F 01 • **[M]** ↓ **F 02** ↓ **F 41** ↓ **F 42**

How many elements belong to the set

$$T(n) = \{(i,k) \in \mathbf{Z}^2 : i \geq 0,\ k \geq 0,\ i + k < n\}?$$

F 02 • **[M]** ↑ **F 01** ↓ **F 41** ↓ **F 42**

How many elements belong to the set

$$U(m,n) = \{(i,k) \in \mathbf{Z}^2 : i \geq 0,\ k \geq 0,\ m < i + k < n\}?$$

F 03 • **[M]** ↓ **F 04** ↓ **F 45**

Let n, m, and k be integers. Denote $s_n^{(k)} = \sum_{m=1}^{n} m^k$. Express $s_n^{(1)}$ in a closed form. Prove your solution by induction.

Hint: This is a well-known formula, derived from the observation that the sums of the first and last member, second and second last member, etc., are all equal to $n+1$ and the number of these sums is $n/2$.
An expression for a quantity $f(n)$ is in closed form if we can compute it using a fixed number of 'well-known' standard operations which are independent of n.

F 04 • **[M]** ↑ **F 03** ↓ **F 07**

The reasoning of Example F 03 cannot be generalized for $k > 1$. Is there another way to reach this result? Consider the equation $s_{n+1}^{(1)} - s_n^{(1)} = n + 1$ and try to find $s_n^{(1)}$ as a second order polynomial of the variable n.

F 05 •

Prove the following equality by induction:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2,$$

i.e. $s_n^{(3)} = (s_n^{(1)})^2$.

F 06 • **[M]** **↑ F 03 ↑ F 04**

Example F 05 yields a closed form expression for $s_n^{(3)}$. Can you use the method shown in example F 04 to reach the same result?

Hint: Assume that $s_n^{(3)} = an^4 + bn^3 + cn^2 + dn + e$.

F 07 • **[M]** **↑ F 04 ↓ F 45**

Express $s_n^{(2)}$ and $s_n^{(4)}$ in a closed form.

Hint: Assume that $s_n^{(2)}$ and $s_n^{(4)}$, respectively are a cubic and a quintic polynomial and follow the solution pattern in Example F 04.

F 08 •• **↑ F 04 ↑ F 06 ↑ F 07 ↓ F 21 ↓ F 22**

Use previous results to obtain the closed form formulae for

$$1^k + 3^k + 5^k + \cdots + (2n - 1)^k$$

with $k = 1, 2, 3, 4$.

F 09 ●● ↑ F 08 ↓ F 15

The result of F 08 for $k = 1$ shows that the square of any positive integer can be expressed as a sum of consecutive odd integers. Find a method to calculate the square root of any positive number to arbitrary precision using only additions.

F 10 ●● ↑ F 00 ↑ F 07 ↓ F 18

Find a closed form for $S(n) = \sum_{0 \le i \le k \le n}(i + k)$.

Hint: Write S as a double sum and observe that $\sum_{i=0}^{k}(i + k) = \frac{3}{2}(k^2 + k)$.

F 11 ●● ↑ F 03 ↑ F 04

Express $s_n = \sum_{k=1}^{n} k(-1)^{k(k+1)/2}$ and $q_n = \sum_{k=1}^{n} k^2(-1)^{k(k+1)/2}$ in a closed form.

Hint: Why can't the method of F 07 be used? Observe that the sum of four consecutive summands in s_n equals 4. And for q_n?

F 12 ●

Show that for complex values of a_k

$$\sum_{k=1}^{n}\left(a_k - \frac{1}{n}\sum_{j=1}^{n} a_j\right)^2 = \sum_{k=1}^{n} a_k^2 - \frac{1}{n}\left(\sum_{j=1}^{n} a_j\right)^2.$$

Hint: Try direct calculation of the left-hand side.

F 13 •

Prove that for arbitrary complex values of a_k

$$\sum_{k=1}^{n} a_k = na_n - \sum_{k=1}^{n-1} k(a_{k+1} - a_k).$$

Hint: Use induction on n.

F 14 • ↑ **F 12**

Prove that a nontrivial quadratic form in two variables is a full square, i.e. there exist real constants α, β such that

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = (\alpha x + \beta y)^2$$

iff its coefficients a_{ik} satisfy the condition $a_{11}a_{22} - (a_{12})^2 = 0$.
Find the conditions ensuring that a quadratic form in n variables is a full square.

Hint: Find the matrix of the quadratic form $(\sum_{i=1}^{n} a_i x_i)^2$ and determine its rank.

F 15 •• ↑ **F 03** ↑ **F 04**

Every positive integer can be expressed as a sum of a certain number of successive positive integers, e.g. as in $15 = 15$, $15 = 7 + 8$, $15 = 4 + 5 + 6$, $15 = 1 + 2 + 3 + 4 + 5$. For a given integer, find a way to determine the number of these sums and the sums themselves. Find examples of integers for which only the trivial decomposition $n = n$ exists.

Hint: Write $n = m + (m + 1) + \cdots + (m + k)$ and deduce that
$2n = (k + 1)(2m + k)$.

F 16 ●●

Assume that m and n are positive integers.
1. Find the number of decompositions of m into n positive integer summands, i.e. $m = x_1 + x_2 + \cdots + x_n$.
2. Solve the previous problem for nonnegative summands.

Hint: 1. Denote $y_k = x_1 + x_2 + \cdots, +x_k, k = 1, 2, \ldots, n$ and note that $1 \leq y_1 < y_2 < \cdots < y_{n-1} \leq m - 1$.
The number of possible choices of such y_k equals to the number of decompositions of m.
2. Apply the previous result for $x_i - 1$.
Notice that the number of decompositions we are asking for is equal to the number of summands in $(a_1 + a_2 + \cdots + a_n)^m$.

F 17 ●●

Prove that harmonic numbers H_n satisfy

$$\frac{1}{2}\left(\lfloor \log_2 n \rfloor + 1\right) < H_n \leq \lfloor \log_2 n \rfloor + 1$$

and deduce that $\lim_{n\to\infty} H_n = +\infty$.

Hint: Start with $n = 2^k$ and note that each group indicated by parentheses in $(1) + (1/2) + (1/3 + 1/4) + (1/5 + \cdots + 1/8) + (1/9 + \cdots + 1/16) + \cdots$ contributes to the sum by a quantity $\geq \frac{1}{2}$.

F 18 ●● ↑ **F 17**

For harmonic numbers H_n obtain the estimate

$$\log(n+1) < H_n < 1 + \log n \quad \text{for } n > 1.$$

Compare this estimate with the one in F 17.

Hint: Consider the upper and lower integral sums for the integral $\int_1^n \frac{1}{x}\,dx$ with partition points at the integer values of x.

4

F 19 ●●

For harmonic numbers H_n no closed form is known. However, some of their properties can be formulated. Prove e.g. the following statements:

$$\sum_{k=1}^{n-1} H_k = nH_n - n \quad \text{and} \quad \sum_{k=1}^{n-1} kH_k = \frac{n(n-1)}{2} H_n - \frac{n(n-1)}{4}.$$

Hint: Use induction with respect to n.

F 20 ●● ↑ **F 19**

Express in terms of H_n the sum $\sum_{k=1}^{n} \sum_{i=1}^{k-1} \frac{k}{k-i}$.

Hint: Change the order of summation and notice that $\frac{k}{k-i} = 1 + \frac{i}{k-i}$.

F 21 ●● ↓ **F 22**

The following theorem can be applied to the partial sums of some special sequences. Let the sequence $u(n)$ satisfy the equation

$$u(n+k) + a_1 u(n+k-1) + \cdots + a_k u(n) = 0$$

for a fixed $k \geq 1$ and for all $n \geq 0$. Then the partial sums $s(n) = u(1) + u(2) + \cdots + u(n)$ satisfy the equation

$$s(n+k+1) - (1-a_1)s(n+k) - (a_1 - a_2)s(n+k-1)$$
$$- \cdots - (a_{k-1} - a_k)s(n+1) - a_k s(n) = 0$$

for $n \geq k-1$. Can you prove this theorem?

Hint: Verify it by direct substitution. As an example you may consider the Fibonacci sequence.

F 22 ••

Define the first difference Δ^1 of a sequence $u(n)$ as

$$\Delta^1 u(n) = u(n+1) - u(n).$$

The second difference is $\Delta^2 u(n) = \Delta^1 u(n+1) - \Delta^1 u(n)$. Can you define the concept of the k-th difference of a sequence $u(n)$? Express it in terms of the values $u(n)$.

F 23 • ↑ **F 22**

Show that the sequence $u(n) = n^k$ satisfies the equation $\Delta^{k+1} n^k = 0$, i.e.

$$u(n+k+1) - \binom{k+1}{1} u(n+k) + \binom{k+1}{2} u(n+k-1) + \cdots + (-1)^k \binom{k+1}{k+1} u(n) = 0.$$

Hint: Use induction on k, recalling that $\Delta^{k+1} n^k = \Delta^k(\Delta n^k)$. The rest follows by direct calculation. Recall that the $(n+1)$-st derivative of x^n equals zero.

F 24 • ↑ **F 22** ↓ **F 26**

Notice the following similarities with basic formulae of differential calculus.
Prove that

$$\Delta^1(u(n) + v(n)) = \Delta^1 u(n) + \Delta^1 v(n),$$

$$\Delta^1 u(n)v(n) = u(n+1)\Delta^1 v(n) + v(n)\Delta^1 u(n),$$

$$\Delta^1\left(\frac{u(n)}{v(n)}\right) = \frac{v(n+1)\Delta^1 u(n) - u(n+1)\Delta^1 v(n)}{v(n+1)v(n)}.$$

F 25 •••[M] ↑ F 04 ↑ F 05 ↑ F 06 ↑ F 07

Prove by induction on n for a fixed k that

$$\binom{k+1}{1}s_n^{(1)} + \binom{k+1}{2}s_n^{(2)} + \cdots + \binom{k+1}{k}s_n^{(k)}$$
$$= (n+1)^{k+1} - (n+1).$$

Use this formula for successive calculation of $s_n^{(k)}$, $k = 1, 2, \ldots$.

Hint: Use the formula $s_n^k = s_{n-1}^k + n^k$ and the following result

```
Sum[n^i Binomial[k + 1, i], {i, 1, k}] =
-1 - n^(1 + k) + (1 + n)^k + n (1 + n)^k
```

F 26 •• ↑ F 24

Let $u(n)$, $v(n)$ be sequences. Prove the following formula of 'summing by parts':

$$\sum_{n=0}^{N} u(n)\Delta^1 v(n) = u(N)v(N+1) - u(0)v(0) - \sum_{n=0}^{N-1} v(n+1)\Delta^1 u(n).$$

(Recall the formula for differentiation of a product and integration by parts.)

F 30 •[M] ↓ F 31 ↓ F 32 ↓ F 33 ↓ F 34 ↓ F 51 ↓ F 52 ↓ F 54

Consider $\sum_{k=0}^{n} q^k$, $q \in \mathbf{C}$. This sum is known to be $\frac{q^{n+1}-1}{q-1}$ for $q \neq 1$. Can you prove that it is correct? Can you derive this result? What about $q = 1$?

F 31 •[M] ↑ F 04 ↑ F 26 ↑ F 30

Suppose that q is a real variable. Using derivatives with respect to q and other computational tools try to express finite sums like $\sum_{k=1}^{n} kq^k$, $\sum_{k=1}^{n} k^2q^k$.

Hint: Taking q as a variable the function $\sum_{k=1}^{n} kq^{k-1}$ can be considered to be the derivative of $\sum_{k=1}^{n} q^k$.

F 32 ••• **↑ F 26**

Solve F 31 without using derivatives.

Hint: Try to derive a difference equation for $S_n = \sum_{k=1}^{n} kq^k$ as follows:

$$S_{n+1} = \sum_{k=1}^{n}(k+1)q^{k+1} + q = \sum_{k=1}^{n} kq^{k+1} + \sum_{k=1}^{n} q^{k+1} + q$$
$$= qS_n + q^2\frac{1-q^n}{1-q} + q.$$

Alternatively, use the result of F 26.

F 33 •• **↑ F 30**

Suppose that q is a square matrix of order m. Can you find a formula expressing $\sigma_n = \sum_{k=0}^{n} q^k$ and conditions under which the formula holds true?

F 34 •• **[M]** **↑ F 30**

Use F 30 with $q = e^{\mathbf{i}x}$, $\mathbf{i}^2 = -1$ to derive a formula for

$$\sum_{k=0}^{n} \sin x \quad \text{and} \quad \sum_{k=0}^{n} \cos x.$$

4

F 35 • [M] ↑ F 31 ↑ F 34

Find the unknown angle x in the equations

$$\sin x + \sin 2x + \cdots + \sin nx = 0, \quad \text{and}$$
$$\cos x + \cos 2x + \cdots + \cos nx = -1.$$

F 36 • ↑ F 34 ↑ F 35

Find the smallest positive zero of $f(x) = \sin x + \sin 3x + \cdots + \sin(2n+1)x$ and prove that it is a local minimum of $f(x)$.

F 37 •• ↓ F 38

Any positive rational number p/q, $p < q$, can be expressed as a finite sum of fractions all numerators of which equal 1 and all denominators are mutually distinct. (E.g. $3/5 = 1/2 + 1/10$.) Give a proof of this statement by constructing an algorithm for finding these fractions.

This example goes back to the ancient Egyptians. They had special symbols for these fractions but no symbols for other rational numbers.

F 38 • ↑ F 37

Show that the finite sum in F 37 is not unique.

F 40 •

Points which have integer coordinates in the Cartesian coordinate system are called integer points. Find conditions for a triangle with vertices at integer points to be right-angled.

Hint: It is no loss of generality to assume that the vertex (x_1, y_1) at the right angle is at the origin.

F 41 • **↑ F 40 ↓ F 42**

Find the number of integer points lying on a segment (including end points) with the given integer end points in a plane. How would you characterize a segment containing no integer points except its endpoints?

Hint: It is sufficient to consider segments with endpoints $(0, 0)$ and (i, k). Why?

F 42 • **↓ F 43 ↓ F 44**

Let $x_i, y_i, i = 1, 2$ be integers. Find the number of integer points inside the rectangle with vertices (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2).

F 43 • **↑ F 01 ↑ F 02**

Let $x_i, y_i, i = 1, 2$ be integers. Find the number of integer points located inside the right-angled triangle with vertices (x_1, y_1), (x_1, y_2), (x_2, y_1).

F 44 • **[M]** **↑ F 01 ↑ F 02 ↑ F 41 ↑ F 43**

Let $x_i, y_i, i = 1, 2$ be integers. Find the number of integer points located inside the trapezium with vertices $(x_1, 0)$, (x_1, y_1), (x_2, y_2), $(x_2, 0)$.

Hint: Decompose the trapezium into a rectangle and a triangle. Take into account the number of points on the boundary of the two areas. You may also analyze and verify the approach using the MATHEMATICA® command

```
s[x_, y_, xx_, yy_] :=
Sum[1, {i, x + 1, xx - 1},
{k, 1, y + Floor[(yy - y)/(xx - x) (i - x)] - 1}]
```

with chosen numerical values of the variables.

F 45 •• ↑ F 41 ↑ F 42 ↑ F 43 ↑ F 44

Find the number of integer points located inside a triangle with vertices given by three integer points.

Hint: By addition of at most three right-angled triangles the given one can be enlarged into a rectangle with sides parallel to the axes. Each of the added triangles is right-angled and therefore it becomes sufficient to use results on rectangles and right-angled triangles.

Examples F 42 – F 45 are special cases of Pick's theorem which gives a relation between the area and number of interior integer points of a 'simple' polygon with integer vertices. This theorem was proved in 1899 and later found numerous generalizations.

F 46 •• ↑ F 40 ↓ F 47

Prove that no regular triangle or hexagon can have integer vertices.

Hint: We may assume the vertices at points $(0,0)$, (m,n), (p,q). Since $e^{\pi\mathbf{i}/3} = (m + \mathbf{i}n)/(p + \mathbf{i}q)$, both $\sin(\pi/3)$ and $\cos(\pi/3)$ must be rational. Similarly for the hexagon.

F 47 ••• PF 47 I 30 ↑ F 46

Prove that the only regular polygon with all $N = 2^k$ vertices at integer points is a square.

Hint: We may assume the three neighboring vertices at points $(0,0)$, (k,m), (p,q). Since $\exp(\mathbf{i}\pi(1-2/N)) = (k+\mathbf{i}m)/(p+\mathbf{i}q)$, both $\sin(2\pi/N)$ and $\cos(2\pi/N\delta)$ must be rational.

F 48 ● ● ● **[M]**

Find, estimate or give an algorithm to evaluate the number of integer points located inside a circle of radius r with center at the origin. Can you give a reasonable estimate of the growth of this number?

F 48 is one of the oldest problems of discrete geometry. It was investigated by Gauss who proved that the number p of points can be estimated by

$$p = \pi r^2 + O(r), \quad r \to \infty.$$

It is conjectured that $p = \pi r^2 + O(r^s)$ for all $s > 1/2$. The best proven result so far is with $s > 24/37$. In 1999 it was found that the number N of points in a closed circle (i.e. including its boundary) of radius r is

$$N(r) = 1 + \sum_{i=1}^{r^2} (-1)^{i-1} \left\lfloor \frac{r^2}{2i-1} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

F 51 ● **[M]**

For a given polynomial $p(x)$ of degree n find the coefficients a_i in $p(x) = \sum_{i=0}^{n} a_i (x-q)^i$ for any given number q.

Hint: Use Horner's scheme.

F 52 ● **[M]** ↑ **F 51**

1. An integer n in the decimal system is known to be divisible by 9 if the sum of its digits is divisible by 9. Prove this statement.
2. Consider positional systems with the base b (b is a positive integer). Find the rules of divisibility of $n_{[b]}$ by the numbers $b-1$ and $b+1$. Is the found rule valid for all nontrivial divisors of $b-1$?
3. Which of the numbers $10\ldots0131$ with q zeros in the middle and $11\ldots1131$ with k ones in the middle is divisible by 11?

Hint: A (decimal) integer is divisible by q if the coefficient with the zero power of q in its expansion in powers of q is divisible by q.

F 53 • ↑ **F 52**

Can any integer be represented in a positional number system of base -2 with digits $-1, 0, 1$? If yes, how can the representation be found?

F 54 • **[M]** ↑ **F 30**

Consider a savings account in a bank with the (compound) interest rate of $p\%$ p.a. With an initial amount A_0, at the end of the n-th year the balance of the account will be $A_n = A_0(1 + p/100)^n$.

1. Find the balance if $A_0 = 0$ and at the beginning of each year the amount A_k, $k = 1, 2, \ldots, n$ is added to the account.
2. How would you calculate the balance for a fixed day of the k-th year? Some banks use the formula $A_d = A_0(1 + p/36000)^d$, where d is the number of elapsed days. Is that correct?
3. Suggest a continuous model and compare it with the methods described above.

F 55 • ↑ **F 54**

Describe the balance of a savings account after n years when an amount B is withdrawn from this account after q years. (Consider the account as in F 54 with an initial balance A_0.)

F 56 • **[M]** ↑ **F 54**

Optimists say that a savings account with an interest rate of $p\%$ p.a. will double the initial investment in $70/p$ years. Pessimists say that the prices will double in $70/r$ years when an $r\%$ inflation rate can be predicted. Can you give some reasons for both two opinions?

Supplementary Material

Definitions

The equation

$$a_0u(n+k)+a_1u(n+k-1)+a_2u(n+k-2)+\cdots+a_ku(n)=0$$

for the unknown sequence $\{u(n)\}_{n=0}^{\infty}$ is called a homogeneous linear difference equation with constant coefficients, a_i are its coefficients, k its order. The polynomial $p(\lambda)=\sum_{i=0}^{k}a_i\lambda^{k-i}$ is called the characteristic polynomial of this equation.

The sum $\sum_{i,k=1}^{n}a_{ik}x_ix_k$ with $a_{ik}=a_{ki}$ is called a quadratic form in n indeterminates (variables) x_i, a_{ik} are its coefficients , and the matrix Q with the element a_{ik} in its i-th row and k-th column is called the matrix of the quadratic form.
The quadratic form is called nontrivial if at least one of its coefficients is nonzero.
The quadratic form is of rank $m\le n$ iff for its matrix Q there exists a nonzero subdeterminant of order m and all its subdeterminants of higher order equal zero.

The sum $\sum_{i=1}^{n}\frac{1}{i}$ is called the harmonic number H_n.

Theorems

A

Let P be a polygon with integer vertices such that its contour does not cross itself. Denote by $I(P)$ the number of interior integer points , by $B(P)$ the number of integer points on its boundary and by $A(P)$ the area of the polygon. Then

$$A(P)=I(P)+B(P)/2-1.$$

4

B

If λ_i, $i = 1, 2, \ldots, k$, are mutually distinct zeros of the characteristic polynomial of a homogeneous linear difference equation of order k with constant coefficients, then all sequences $u(n)$ satisfying the equation are given by

$$u(n) = \sum_{i=1}^{k} c_i \lambda_i^n, \quad n = 1, 2, \ldots$$

where the coefficients c_i are uniquely determined by the values $u(0), u(1), \ldots, u \times (n-1)$.

C

A matrix A of type $n \times n$ is symmetric iff there is an orthonormal matrix C and a diagonal matrix D such that $A = C^T DC$.

D

Write the polynomial $P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ in the form

$$P(x) = ((\ldots(a_0 x + a_1)x + a_2) + \cdots + a_{n-1})x + a_n.$$

For any h the sequence $\{b\}$ defined by

$$b_0 = a_0, \qquad b_{i+1} = b_i h + a_{i+1}, \quad i = 0, 1, \ldots, n-1 \quad (*)$$

yields a convenient way to evaluate $P(x)$ for any given value $x = h$ since $b_n = P(h)$. Moreover, for the polynomial $Q(x) = b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-1}$ we have $P(x) = (x-h)Q(x) + b_n$ and therefore $P'(h) = Q(h)$.
In case $P(h) = 0$ the polynomial $Q(x) = \frac{P(x)}{x-h}$.

Corollary:

1. The sequence $\{b\}$ yields a convenient way to calculate the value $P(h)$ (Horner's scheme).
2. Repeated application of the procedure yields the value of $P'(h)$.
3. For a polynomial $P(x) = p_0(x)$ and for arbitrary values c_i form a sequence $P_{i+1}(x) = (x-h)P_i(x) + c_i$. Then

$$P_{i+1}^{(n)}(h) = n! P_i(h),$$

i.e. repeated application of procedure $(*)$ yields the Taylor coefficients of the polynomial P with the center h.

Plans of Solution

PF 47

1. We may assume that the three neighboring vertices are at points $(0,0)$, (m,n), (p,q). Since

$$e^{2\pi \mathbf{i}/N} = \frac{m+\mathbf{i}n}{p+\mathbf{i}q},$$

both $\sin(2\pi/N)$ and $\cos(2\pi/N)$ must be rational.
2. Note that $\cos(2\pi/2^3), \cos(2\pi/2^4), \ldots$ is irrational.
   Use the formula $\cos(\alpha/2) = \sqrt{\frac{1+\cos\alpha}{2}}$.

Further References

Graham R.L., Knuth D.E., Patashnik O., Concrete Mathematics, Addison-Wesley, New York, 1989.

Petrovšek M., Wilf H.S., Zeilberger D., A = B, http://www.math.upenn.edu/~wilf/AeqB.html.

For more information on difference equations see standard monographs. One of the latest where additional references can be found is

Kelley W.G., Peterson A.C., Difference Equations, An Introduction with Applications, Academic Press, New York, 2001.

Examples F 40–50 are connected to number theory and discrete geometry; see

Vinogradov L.M., An Introduction to the Number Theory, Pergamon, New York, 1955.

K. Voss, Discrete Images, Objects and Functions in Z^n, Springer, Berlin, 1993.

Inequalities 5

Quantitative descriptions of physical, technical and economical phenomena include quantities which are inherently positive. Their mathematical models are adequate only if this fact is reflected by the mathematical structure involved. Positivity, i.e. a special inequality, is therefore an important issue in analysis and applications.

On the other hand, the square of any real number is nonnegative. This fact alone is both a source and a method of proof for many important inequalities, which are widely used in analysis and its applications.

In this chapter you will find some inequalities which are often used in mathematical reasoning and their knowledge is indispensable. Also methods of deriving new inequalities and estimates are included. Last but not least, applications of inequalities in solving mathematical problems arising outside mathematics are included.

The solution of problems marked [M] and many others can be made easier by using software packages of S. Wolfram's MATHEMATICA®, although only very few of them can be solved directly. The user is strongly advised to learn the syntax and semantics of some of the commands. In this chapter the following commands in MATHEMATICA® (and related ones) may help:

`Plot, ListPlot, Plot3D, ContourPlot, LinearProgramming, FindRoot`

Similarly in Maple®:

`plot, plot3d, plots[contourplot], Optimization[LPSolve], solve, Student[NumericalAnalysis][Roots]`

In most of the examples numerical experiments may give a starting point for reasoning or verify initial conjectures.

Explanation of mathematical terms and concepts can also be found on the Internet, e.g. at www.mathworld.wolfram.com or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://eom.springer.de.

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5

Suggestions

- Indispensable even in elementary calculus are the results of E 01, E 02, E 04, E 05 and useful methods are presented in E 10, E 13, E 36. Continue further along the downward arrows.
- Those who are research inclined will find interesting stimuli in E 16, E 19, E 23, E 24, E 28, E 34.
- Teachers could use E 18, E 25, E 29, E 30, E 40 to enhance understanding and for broader context.
- All readers are strongly encouraged to modify, generalize or simplify the formulated problems, to find alternative formulations, formulate and solve their own examples and compare the context of these problems to the given ones.

Problems

E 01 • $\downarrow$ **I 63** $\downarrow$ **E 03** $\downarrow$ **E 04** $\downarrow$ **E 19** $\downarrow$ **E 31** $\downarrow$ **E 32** $\downarrow$ **E 33** $\downarrow$ **E 34** $\downarrow$ **E 35**

Show that the arithmetic mean of any two positive numbers is never smaller than their geometric mean, i.e. that

$$a + b \geq 2\sqrt{ab}.$$

E 02 • $\uparrow$ **E 01**

Prove the following inequality for any two real numbers a, b and find its geometric interpretation:

$$\sqrt{a^2 + b^2} \leq |a| + |b|.$$

E 03 • $\uparrow$ **E 01** $\downarrow$ **E 32**

Prove that the harmonic mean of any two positive numbers is never greater then their geometric mean, i.e. that

$$\frac{2ab}{a+b} \leq \sqrt{ab}.$$

E 04 •

Find the necessary and sufficient conditions concerning the real numbers a, b which would ensure equality in the inequalities of E 01, E 02, E 03.

E 05 •• ↑ **E 01** ↑ **E 02**

Let a, b be complex numbers. Find a proof of the triangular inequality

$$||a| - |b|| \leq |a + b| \leq |a| + |b|.$$

Find the necessary and sufficient conditions for a, b such that

$$|a + b| = |a| + |b|.$$

Prove also the following generalization:

$$\left|\sum_{i=0}^{n} a_i\right| \leq \sum_{i=0}^{n} |a_i|.$$

Hint: For the right-hand side inequality consider $|a + b|^2$, use the fact that $|a|^2 = a\bar{a}$ and recall that $\operatorname{Re} z \leq |z|$ for any complex z, with ($\bar{a} = \text{Conjugate}[a]$). For the left-hand side inequality put $b - a$ instead of b.

This is probably the most important and most frequently used inequality. The number of its generalizations as well as its use in proofs, definitions and applications can hardly be overestimated.

E 06 •

Find conditions for two positive numbers a, b ensuring that $a^b < b^a$.

Hint: Try to reduce the problem to investigation of the function

$$f(x) = \frac{\log x}{x}.$$

E 07 ●● ↑ E 06

It can be said that the exponential function of a single real variable grows faster than any power of the variable. Give a precise formulation of this statement and prove it.

E 08 ●●

Let $a_1, a_2, \ldots, a_n, b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ be two sequences of real numbers and let

$$s_k = a_1 + a_2 + \cdots + a_k, \qquad m = \min s_k, \qquad M = \max s_k.$$

Then

$$mb_1 \leq a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq Mb_1$$

(Abel's inequality). Find a proof.

Hint: Express the sum $\sum_{k=1}^{n} a_kb_k$ in terms of b_k, s_k.

E 09 ● **[M]**

Characterize the set A composed of points $(x, y) \in \mathbf{R}^2$ satisfying the inequalities:

1. $|x| + |y| \leq 1$,
2. $\max(|x|, |y|) \leq 1$.

Characterize the set A composed of points $(x, y, z) \in \mathbf{R}^3$ satisfying the inequalities:

3. $|x| + |y| + |z| \leq 1$,
4. $\max(|x|, |y|, |z|) \leq 1$.

Hint: Note the symmetry $(x, y) \in A \Rightarrow (\pm x, \pm y) \in A$ and similarly for $\mathbf{R}^3$.

E 10 ●● ↓ E 11

Prove the inequality

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

Hint: Consider the nonnegative quantity $\sum_{i=1}^{n}(x_i u + y_i)^2$ in a variable u and look at the discriminant.

E 11 ● ↑ E 10

Let f, g be square integrable functions on (a, b). Prove that

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx.$$

Hint: Consider $\int_a^b (f(x)u + g(x))^2 dx$ for a variable u.

Inequalities in E 10, E 11 are special cases of the important inequality named after Cauchy, Schwarz and Bunjakovskij, which is widely used and cited under one or two of these names.

E 12 ●●

Let $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n)$ be n-tuples of real numbers such that both are increasing or both are decreasing. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \leq \left(\frac{1}{n}\sum_{k=1}^{n} a_k b_k\right)$$

(Chebyshev's inequality).

Hint: Denote $\sum a = \sum_{k=1}^{n} a_k$ and consider $n\sum ab - \sum a \sum b$ as a double sum which can be expressed in terms of $(a_k - a_j)(b_k - b_j) \geq 0$.

5

E 13 • **[M]** ↓ **E 15** ↓ **E 16** ↓ **E 17** ↓ **E 18**

For $x \in [0, \pi/2]$ prove that $\frac{2}{\pi}x \leq \sin x \leq x$ (Jordan's inequality).

Hint: Analyze the behavior of $f(x) = x - \sin x$ and $g(x) = \sin x - \frac{2}{\pi}x$.

E 14 •

Suppose that two functions f, g are defined on the same interval T and $f(x) \leq g(x)$ for all $x \in T$. Prove or disprove the following statements:

(a) $f'(x) \leq g'(x)$,
(b) $\int_y^x f(x) \leq \int_y^x g(x)$, $y < x$.

E 15 • **[M]** ↑ **E 13**

Show that

$$x - \frac{x^3}{6} \leq \sin x \quad \text{and} \quad 1 - \frac{x^2}{2} \leq \cos x \quad \text{for } 0 \leq x \leq \pi/2.$$

Hint: Proceed with a direct proof similar to the one in E 13.

E 16 •• **[M]** ↑ **E 13**

For $x \geq 0$ and $p \geq 1$ prove that $(1+x)^p \geq 1 + px$. Is it valid in other cases?

Hint: The proof is easy for integer values of p. Otherwise use the method of E 13.

E 17 • **[M]** ↑ **E 13**

Find all values of x for which the inequality $\frac{1}{1+x^2} \geq e^{-x^2}$ holds true.

Hint: It may help to plot the two functions.

E 18 ●● [M] ↑ E 13 ↓ E 26

Find a proof of the following inequalities provided that $x > 0$:

$$x^a - ax + a - 1 \le 0 \quad \text{if } 0 < a < 1,$$
$$x^a - ax + a - 1 \ge 0 \quad \text{for other values of } a.$$

Use this result to derive the inequality

$$x^a y^{1-a} \le ax + (1-a)y, \quad 0 < a < 1,$$

and show that the inequality in E 01 is its special case.

Hint: Consider the derivative of the left-hand side of the inequalities. To prove the third one use x/y instead of x in the first inequality.

E 19 ●●

Let a nonnegative function f have a continuous derivative on the interval $[a, b]$ and let $f(a) = f(b) = 0$. Then there must exist a number $\xi \in [a, b]$ such that

$$f'(\xi) \ge \frac{4}{(b-a)^2} \int_a^b f(x)dx.$$

Find a proof.

Hint: There exists a ξ such that $|f'(\xi)| = \max_{x\in[a,b]} |f'(\xi)| = M$. Then $f(x) \le M(x-a)$ for all $a \le x \le \frac{a+b}{2}$ and $f(x) \le M(b-x)$ for all $\frac{a+b}{2} \le x \le b$ and the result follows.
Plot a graph representing these inequalities.

E 20 ●

Assume that in E 19 the function f is the velocity of motion and find a physical interpretation of the inequality therein.

E 21 • ↓ **E 22**

Give a geometric interpretation of a convex function in terms of its graph and chords passing through the two points $(x_1, f(x_1))$, $(x_2, f(x_2))$.

Hint: In the definition of a convex function consider first the geometric meaning of equality.

E 22 • ↑ **E 21** ↓ **E 25**

Suppose that f and g are convex functions on a closed interval $[a, b]$. For each of the functions h_i described below determine whether it is convex:

$$h_1 = f + g, \qquad h_2 = fg, \qquad h_3 = \int_a^x f(t)dt, \qquad h_4 = f(g),$$

provided that the composition of f and g is defined. Find examples and counterexamples.

E 23 •• ↑ **E 14**

Let f and g be positive functions with continuous and nonnegative derivatives on $[0, b]$ and let $f(0) = 0$. Then for $0 < a \leq b$

$$f(a)g(b) \leq \int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx.$$

Prove this statement.

Hint: Integrate by parts the first integral on the right-hand side.

E 24 • **M 24**

Let f be a continuous and strictly increasing function on $[0, c], c > 0$. If $f(0) = 0, a \in [0, c], b \in [0, f(c)]$ then

$$\int_0^a f(x)dx + \int_0^b f_{-1}(x)dx \geq ab,$$

where f_{-1} is the inverse function of f.

Verify this statement (Young's inequality) by its geometric interpretation.

E 25 •• **↑ E 18 ↑ E 21 ↓ E 26 ↓ E 33 ↓ E 34 ↓ E 35**

Suppose that the function f is convex on a closed interval $[a, b]$ and that $a \leq x_i \leq b$, $i = 1, 2, \dots, n$, $\alpha_i > 0$, $\sum \alpha_i = 1$. Then

$$\sum_{i=1}^{n} \alpha_i f(x_i) \geq f\left(\sum_{i=1}^{n} \alpha_i x_i\right).$$

Find a proof. Formulate a similar result for concave functions.

Hint: Use induction over n. Assuming the validity for n, write

$$\sum_{i=1}^{n+1} \alpha_i x_i = \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + \hat{\alpha}\hat{x},$$

where

$$\hat{\alpha} = \alpha_n + \alpha_{n+1} \quad \text{and} \quad \hat{x} = \frac{\alpha_n x_n}{\hat{\alpha}} + \frac{\alpha_{n+1} x_{n+1}}{\hat{\alpha}}.$$

5

E 26 ●● **↑ E 01 ↑ E 25**

Prove that the inequality

$$\log\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \geq \frac{1}{n}\sum_{i=1}^{n}\log x_i$$

for $x_i > 0$ follows from concavity of log.
Deduce from the inequality above that the arithmetic mean of n positive numbers is never smaller than their geometric mean, i.e. that

$$\frac{1}{n}\sum_{i=1}^{n}x_i \geq \left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}} \quad \text{for } x_i > 0.$$

Hint: The first part can be accomplished by induction.

E 27 ●● **↑ E 01 ↑ E 18 ↑ E 26**

The concept of convex functions enables to find and prove a number of important inequalities, e.g. as in the following statements:
1. For nonnegative values of x, y, for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ there is

$$x^{\frac{1}{p}}y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q} \quad \text{or} \quad xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

2. Prove that

$$\log\left(\frac{\exp(-x)}{p} + \frac{\exp(-y)}{q}\right) + \frac{x}{p} + \frac{y}{q} \geq 0$$

for all real x, y and deduce a similar inequality with n summands.

Hint: 1. Put $x = \exp(\alpha)$, $y = \exp(\beta)$ and use the convexity of the exponential function.
2. In a more general inequality

$$\log\sum_{i=1}^{n}\frac{\exp(-x_i)}{a_i} + \sum_{i=1}^{n}\frac{x_i}{a_i} \geq 0,$$

with $\sum a_i = 1, a_i > 0$ we may recognize the inequality of E 26 (with a special choice of a_i).

E 28 •

↓ E 29 ↓ E 30

Characterize the set A composed of points (x, y) satisfying the inequalities $2x + 3y \le 3$, $x - y \le 1$, $-2x + y \le 4$.
Find the maximum value of $f(x, y) = x^2 + y^2$ and $g(x, y) = 3x + 4y$ across the set A. Can you change one of the coefficients in the inequalities so that the problem would have no solution?

E 29 ••

↓ PE 29 ↑ E 28

Suppose that two types of products P_1, P_2 can be produced using four types of raw material S_i, $i = 1, 2, 3, 4$, where up to b_i amounts of each can be used. The amount of raw material to be used equals a_{ik} for the i-th product and k-th material. The revenue of a unit of each of P_i equals c_i. We want to organize the production so as to maximize the revenue.
Denoting x_1, x_2 the quantities of production for P_1 and P_2, respectively, formulate the conditions under which the maximum of the function $F = c_1 x_1 + c_2 x_2$ for unknown x_1 and x_2 is to be found.
Find the solution of the problem with the following data:

$$c_1 = 7, \qquad c_2 = 5, \qquad (b_1, b_2, b_3, b_4) = (19, 13, 15, 18),$$
$$(a_{11}, a_{12}, a_{13}, a_{14}) = (2, 2, 0, 3),$$
$$(a_{21}, a_{22}, a_{23}, a_{24}) = (3, 1, 3, 0).$$

E 30 ••

↓ PE 30 ↑ E 28

Assume that we have to distribute from two places A_1, A_2 the amounts a_1 and a_2 tons of some material to three destinations B_i, which demand b_i tons each. The transportation costs per unit from A_i to B_j are known to be c_{ij}. The overall transportation costs are therefore given by

$$F = \sum_{i,j} c_{ij} x_{ij},$$

where x_{ij} denotes the amount to be transported from A_i to B_j. Assume that $a_1 + a_2 = b_1 + b_2 + b_3$. Find the values x_{ij} so as to minimize the function F.
Put $(b_1, b_2, b_3) = (10, 30, 10)$, $(a_1, a_2) = (20, 30)$ and $F = 4x_{11} + 9x_{12} + 3x_{13} + 4x_{21} + 8x_{22} + x_{23}$.

Problems E 28, E 29, E 30 are extremely simplified examples of so called linear programming problems. Using the notation

$$a \geq 0 \Longleftrightarrow a_i \geq 0 \quad \text{for all } i = 1, 2, \ldots, n$$

for n-dimensional vectors, it can be formulated as follows:
Let A be a matrix with n rows and m columns, $m > n$. Among positive solutions of the system of linear inequalities

$$Ax - b \geq 0$$

find the one which gives minimum value to the linear form

$$F = \sum_{i=1}^{m} a_i x_i$$

with given coefficients a_i. Here $x = (x_1, x_2, \ldots, x_m)$, and similarly for b.
These types of problems often occur in planning, e.g. in resource allocation, transport planning where often the dimensions of vectors and matrices are several tens or hundreds. Special methods have been developed for these purposes; the so-called simplex method is the most important.

E 31 •• ↑ **E 01** ↑ **E 25**

If the product of n positive numbers equals 1 then their sum cannot be smaller than n.
Prove this statement.

Hint: Apply E 25 for $f(x) = \exp(x)$.

E 32 • ↑ **E 01** ↑ **E 18** ↑ **E 26** ↑ **E 27**

For $\frac{1}{p} + \frac{1}{q} = 1$ and $a > 0, b > 0$ prove that

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \geq \frac{a}{p} + \frac{b}{q} \quad \text{for } p < 1.$$

Hint: Put $x = a/b$ in a suitable case of E 18.

E 33 ●● ↑ **E 01** ↑ **E 25** ↑ **E 31**

For $\alpha \le 0 \le \beta$ and for positive numbers x_i

$$\left(\frac{x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha}{n}\right)^{\frac{1}{\alpha}} \le \sqrt[n]{x_1 \dots x_n} \le \left(\frac{x_1^\beta + x_2^\beta + \cdots + x_n^\beta}{n}\right)^{\frac{1}{\beta}}.$$

Find a proof.

Hint: Apply the inequality of E 25 for concave function $\exp(\alpha x)$ and for convex function $\exp(\beta x)$.

E 34 ●● ↑ **E 25** ↑ **E 26**

The symbol c_α denotes the mean of order α of n positive numbers x_i, i.e.

$$c_\alpha = \left(\frac{x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha}{n}\right)^{\frac{1}{\alpha}}.$$

Prove that $\alpha \le \beta$ implies $c_\alpha \le c_\beta$.

Hint: Consider $f(t) = \log c_t$ and calculate $\frac{d}{dt} f(t)$ for $t \neq 0$.

E 35 ●● ↑ **E 25** ↑ **E 26**

If c_α is the mean of order α of n positive numbers x_i, then

$$\lim_{\alpha \to 0} c_\alpha = \sqrt[n]{x_1 x_2 \dots x_n}$$

and

$$\lim_{\alpha \to \infty} c_\alpha = \max(x_1 x_2 \dots x_n).$$

5

E 36 •

Prove that

(i) $4x^2 - xy + y^2 > 0$ for all real nonzero values x, y;
(ii) $5x^2 - 4xy + 5y^2 - 12xz - 2yz + 10z^2 > 0$ for all real nonzero values x, y, z.

The left-hand side expressions here are called quadratic forms in two and three variables. Quadratic forms satisfying conditions of positivity for all real values of their variables are called positive definite forms. This concept plays an important role in linear algebra and other branches of mathematics.

E 37 •• ↑ **E 01** ↑ **E 05**

Let the polynomial $f(z) = \sum_{k=0}^{n} a_k z^{n-k}$ have complex coefficients. Then any of its zeros z satisfies the inequality $|z| \le 1 + \max|\frac{a_k}{a_0}|$. Can you prove this estimate?

Hint: If the zero z satisfies $|z| \le 1$ there is nothing to prove. For $|z| > 1$ and $\mu = \max|\frac{a_i}{a_0}|$ use the inequalities

$$\sum_{k=1}^{n} |z^{-k}| \le \sum_{k=1}^{\infty} |z^{-k}| = \frac{1}{|z|-1}$$

and

$$|f(z)| = |a_0/z|^n \left|1 + \sum_{k=1}^{n} (a_k/a_0) z^{-k}\right| \ge \left|\frac{a_0}{z}\right|^n \left(1 - \mu \sum_{k=1}^{n} |z^{-k}|\right).$$

E 38 • ↓ **E 40** ↓ **E 41**

The relation $\prec_F$ in the set of natural numbers is defined as follows: for $n, m \in N$ there is $n \prec_F m$ iff n divides m.
Is this relation a (partial or linear) order?

Hint: It may help to represent graphically the partial order of a given set.

E 39 ••

Let L be the set of all points with integer coordinates in the plane. Define the binary relation $\prec_L$ as follows: $(a, b) \prec_L (x, y)$ iff $(a + b < x + y)$ or $(a + b = x + y$ and $a - b \leq x - y)$. Is this binary relation a linear order?
Consider this binary relation for the set $M \subset L$ consisting of points with nonnegative integer coordinates. Find the minimal element of M with respect to $\prec_L$. Mark each point (m, n) of M by the integer $f(m, n)$ determining its position in the considered ordering. Can you give an explicit expression for $f(m, n)$?

E 40 •

Let X be the collection of all finite sequences of 0's and 1's. We define for $x, y \in X : x \prec_E y$ iff $y = x$ or y extends x. Is $\prec_E$ a partial order?

E 41 ••

Let S be the set of all integer-valued sequences. We define for $a, b \in S$ two relations

$a \prec_P b$ iff $a_n \leq b_n$ for all n

and

$a \prec_A b$ iff the set $\{n : a_n > b_n\}$ is finite.

Show that both $\prec_P, \prec_A$ are partial orders. Let $e_n = (n, n, \ldots)$.

1. Does there exist an element $c \in S$ such that for all n we have $e_n \prec_P c$?
2. Does there exist an element $c \in S$ such that for all n we have $e_n \prec_A c$?

5

Supplementary Material

Definitions

A function f defined on the interval $[a, b]$ is called convex on this interval if the inequality

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y)$$

holds whenever $x, y \in [a, b]$ and $0 \leq \lambda \leq 1$.
The function f is called concave on this interval if

$$f((1-\lambda)x + \lambda y) \geq (1-\lambda) f(x) + \lambda f(y)$$

holds whenever $x, y \in [a, b]$ and $0 \leq \lambda \leq 1$.

For n positive numbers a_i the value

$$c_\alpha = \left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}$$

is called their mean value of order α.
In particular, c_1 is called their arithmetic mean, c_{-1} their harmonic mean.

The expression $\sum_{i,k=1}^n a_{ik} x_i x_k$ is called a quadratic form in n variables $x_1, x_2, \ldots, x_n$. The real numbers $a_{ik} = a_{ki}$ are its coefficients and the matrix $A = [a_{ik}]$ is called its matrix.

The set A is called partially ordered if a binary relation $\prec$, called partial order, is defined satisfying the following conditions:

1. (Reflexivity): $a \prec a$;
2. (Antisymmetry): if $a \prec b$ and $b \prec a$, then $a = b$;
3. (Transitivity): if $a \prec b$ and $b \prec c$, then $a \prec c$.

If such a relation is defined for all pairs of A then the set A is called linearly ordered.

Theorems

A

If the function f is twice differentiable on the interval I, then f is convex on I iff the second derivative of f is nonnegative on I.

Plans of Solution

PE 29

1. The quantities x_i must be nonnegative.
2. For each b_k the inequality $a_{1k}x_1 + a_{2k}x_2 \leq b_k$, $k = 1, 2, 3, 4$, must be respected.
3. Among all solutions of the inequalities $x_1 \geq 0$, $x_2 \geq 0$, $2x_1 + 3x_2 \leq 19$, $2x_1 + x_2 \leq 13$, $3x_2 \leq 15$, $3x_1 \leq 18$ such values of x_1, x_2 are to be found which give the function $F = 7x_1 + 5x_2$ its maximal value.
4. The graphs of the involved straight lines enable to find the solution in this case.

PE 30

1. The following equalities hold:

$$x_{11} + x_{21} = b_1, \qquad x_{12} + x_{22} = b_2, \qquad x_{13} + x_{23} = b_3,$$
$$x_{11} + x_{12} + x_{13} = a_1, \qquad x_{21} + x_{22} + x_{23} = a_2.$$

2. There are six unknowns and the rank of the system is 4. Hence two of the unknowns can be chosen arbitrarily.
3. Rewrite the equations and the function F in terms of these two, say x_{11}, x_{12}, and take into account that all the unknowns must be positive.
4. Obtain a system of six linear inequalities. Choosing x_{11}, x_{12} as the coordinate axes and F as the equation of a plane we obtain the result.

Further References

Systematic treatment of inequalities can be found e.g. in

Beckenbach E.F., Bellman R., Inequalities, Springer, Berlin, 1961.

Hardy G.H., Littlewood J.E., Polya G., Inequalities, Cambridge University Press, London, 1951.

Mitrinovic D.S., Analytic Inequalities, Springer, New York, 1970.

Collocation and Least Squares Methods 6

Building a mathematical models often starts with a pattern, e.g. with an equation or function, which is supposed to characterize the object of investigation or to describe its behavior. This pattern often depends on a number of parameters (coefficients, exponents or other constants, e.g. coefficients of linear combination in a linear space), which have to be determined so the model becomes as close to the known facts as possible. Various methods of calculating unknown parameters are usually much simpler than other methods of designing a mathematical model. A similar problem arises when a simpler form of a given mathematical expression is supposed to exist. This supposed form or template may depend on some parameters which have to be determined.

In this chapter you will learn various methods how to adjust given or chosen mathematical models to actual requirements and actual data. Such adjustment means choice of proper values of some parameters, which usually leads to solution of a set of equations. This may be simple if these equations are linear in the unknown parameters. In other cases no general approach exists and some ad hoc methods have to be applied.

The solution of problems marked [M] and many others can be made easier when using software packages comparable to S. Wolfram's MATHEMATICA® or Maple®, although only very few of them can be solved directly. The user is strongly advised to learn the syntax and semantics of some of the commands. In this chapter the following commands (and related ones) may help:

In MATHEMATICA®

```
Solve, DSolve, RSolve, Fit, Plot, Sum,Inverse,
InterpolatingFunction, InterpolatingPolynomial,
PseudoInverse, PadeApproximant, Series.
```

J. Gregor, J. Tišer, *Discovering Mathematics*,
DOI 10.1007/978-0-85729-064-9_7, © Springer-Verlag London Limited 2011

6

In Maple®

```
solve, fsolve, dsolve (and  ODE Analyzer Assistant
or DEToolsPackage, rsolve, LinearAlgebra[LinearSolve,
MatrixInverse],CurveFitting[PolynomialInterpolation]
(and CurveFitting Assistant),taylor, series.
```

In most of the examples numerical experiments may give a starting point for reasoning or verify initial conjectures. Explanation of mathematical terms and concepts can also be found on the Internet, e.g. at www.mathworld.wolfram.com or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://eom.springer.de.

Suggestions

- The simplest and most used collocation problems are with polynomials. The starting points can be Q 01, Q 02, Q 04, Q 15, Q 17, Q 19.
- More general problems are e.g. the Q 03, Q 05, Q 06, Q 11, Q 12, Q 28, Q 29.
- Application of collocation methods in solving differential equations is illustrated in Q 10, Q 20, Q 21, Q 22, Q 23.
- More involved problems (including some nonlinear ones) are Q 07, Q 08, Q 14, Q 18.

Problems

Q 01 • ↓ **Q 02**

(a) Characterize all polynomials assuming rational values at all rational numbers.
(b) Verify the following statement: A polynomial has integer coefficients iff all values at integers are integers.

Q 02 • [M] ↑ Q 01 ↓ Q 03 ↓ Q 04 ↓ Q 05 ↓ Q 06 ↓ Q 10 ↓ Q 15 ↓ Q 16 ↓ Q 19 ↓ Q 20 ↓ Q 21

Find the coefficients of a polynomial $p(x)$ of lowest possible degree assuming at the points -3, -1, 2, 5 given values 1, 2, 3, 4, respectively, by solving the system of linear equations for the unknown coefficients.
Does this polynomial assume integer (rational) values at all integer (rational) values of its variable?

The polynomial $p(x)$ is called the interpolating polynomial, the $(1, 2, 3, 4)$ are its values, the $(-3, -1, 2, 5)$ are its nodes.

Q 03 • ↓ Q 04 ↓ Q 05 ↓ Q 06

Does the method of Q 02 yield in all cases coefficients of a polynomial of degree n, which assumes given values y_k at given points x_k, $k = 0, 1, 2, \ldots, n$? Is the ordering of the points significant? Is the value $\min_{0\le i,j\le n} |x_i - x_j|$ important?

Hint: Coefficients of the polynomial are the solution of a system of linear equations. The determinant of the system is known as Vandermonde's determinant V. Its evaluation gives the answer to the problem.

Q 04 •• ↑ Q 02 ↓ Q 15

For a cubic polynomial p let the values $p(x_1)$, $p'(x_2)$, $p'(x_3)$, $p''(x_4)$ be given. Can such a polynomial be found for arbitrary mutually distinct nodes x_i?

6

Q 05 • ↓ **Q 06** ↑ **Q 02**

For mutually different numbers x_k, $k = 0, 1, 2, \ldots, n$ put $\omega(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$. Define functions

$$l_i(x) = \frac{\omega(x)}{\omega'(x_i)(x - x_i)}, \quad i = 0, 1, 2, \ldots, n.$$

1. Prove that l_i are polynomials of degree n and

$$l_i(x_k) = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

2. Prove that for a function f defined at x_k the function Ω given as

$$\Omega(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$$

is a polynomial of degree $\le n$ with $\Omega(x_k) = f(x_k)$, i.e. Ω interpolates f at x_k, $k = 0, 1, 2, \ldots, n$.
3. Find conditions such that $\deg \Omega < n$.

Q 06 • **[M]** ↑ **Q 02** ↑ **Q 05**

For a given set of nodes $x_k \in \mathbf{R}$, $k = 0, 1, 2, \ldots, n$ define $\nu_i(x) = \prod_{k=0}^{i-1}(x - x_k)$, $i = 0, 1, 2, \ldots, n$. Recall that $\nu_0(x) = 1$. For given values $y_k = f(x_k)$ find the coefficients α_i in $y_k = \sum_{i=0}^{n} \alpha_i \nu_i(x_k)$. Prove that the polynomial $y(x) = \sum_{i=0}^{n} \alpha_i \nu_i(x)$ satisfies $f(x_k) = y(x_k)$, i.e. that y interpolates f at x_k.

The $l_i(x)$ in Q 05 are called Lagrange polynomials. The $y(x)$ in Q 06 is called Newton's interpolating polynomial. For given data $(x_k, f(x_k))$, $k = 0, 1, 2\ldots, n$ the two polynomials are identical. Both form an orthogonal system on the given set of nodes. The advantage of Newton's polynomials is that if we want to add a node x_{n+1} with given $f(x_{n+1})$ then for to obtain the polynomial Ω all the polynomials l_i have to be recalculated, while for y only one further summand has to be attached. In both cases the formulae can be substantially simplified for systems with equidistant nodes.

Q 07 $\bullet\bullet\bullet$ **[M]** $\uparrow$ **Q 05** $\uparrow$ **Q 06**

A case study: Comparison of interpolating polynomials of various degrees and various systems of nodes. For the function $f(x) = 1 - |x|$, $-1 \le x \le 1$ consider two interpolation schemes:

1. interpolating polynomials for f at equidistant nodes $x_k = \frac{2k}{n} - 1$, $k = 0, 1, 2, \ldots, n$ and
2. interpolating polynomials for f at nodes $x_k = \cos\frac{(2k+1)\pi}{2n+2}$, $k = 0, 1, 2, \ldots, n$.

Compare the two results e.g. for $n = 4, 6, 10$. Find an interpolating polynomial with randomly chosen nodes and compare it with the previous ones.

The inconvenient feature of interpolation with equidistant nodes shown in this example is called Runge's phenomenon (like that of Gibbs phenomena in the theory of Fourier series). The other system of nodes is called Chebyshev nodes. These nodes are the zeros of Chebyshev polynomials $T_n(x) = \cos(n \arccos x)$ and they often give better results.

Q 08 $\bullet\bullet$ $\uparrow$ **Q 05** $\downarrow$ **Q 09**

Show that the system of functions $h_k(x) = \frac{\sin(\sigma x - k\pi)}{\sigma x - k\pi}$, $k \in \mathbf{Z}$ has the following property for all $\sigma > 0$: $h_k(\frac{n\pi}{\sigma}) = \delta_{nk}$. Formulate the corresponding interpolation formula and apply it (for a chosen σ and some values of N) to functions

$$f_N(x) = \begin{cases} x & \text{for } -\frac{N\pi}{\sigma} < x < \frac{N\pi}{\sigma}, \\ 0 & \text{otherwise.} \end{cases}$$

This example is closely connected to the sampling theorem, which plays an important role in information and signal theory. The sampling theorem is a statement that there exists a well-defined class of continuous functions such that each of them is fully characterized by its samples (i.e. values at a countable set of sampling points) provided these sampling points are at predefined distances. The interpolation formula and the sampling theorem for band-limited signals is due to Kotělnikov, Whittaker, Shannon, Nyquist (see http://mathworld.wolfram.com/SamplingTheorem.htm). Note that the functions h_k form an orthogonal system on the set of integers.

Q 09 ●● ↑ Q 05 ↑ Q 08

Let functions $u_k : I \to \mathbf{R}$, where $I \subset \mathbf{R}$ is an interval, be given and numbers x_k, $k = 1, 2, \ldots, n$. Find the conditions imposed upon the functions u_k and the numbers x_k such that there exist coefficients α_{ik} in

$$\varphi_i(x) = \sum_{k=1}^{n} \alpha_{ik} u_k(x), \quad i = 1, 2, \ldots, n$$

so that $\varphi_i(x_k) = \delta_{ik}$. Verify that $\Phi(x) = \sum_{i=1}^{n} f(x_i)\varphi_i(x)$ interpolates the function f at $x_k, k = 1, 2, \ldots, n$.

Q 10 ●● ↑ Q 02

Define as a quasipolynomial the function f with

$$f(t) = p_1(t)e^{\lambda_1 t} + p_2(t)e^{\lambda_2 t} + \cdots + p_n(t)e^{\lambda_n t},$$

where the λ_i are mutually distinct real numbers and the p_i are polynomials.

1. Prove that all derivatives of a quasipolynomial are quasipolynomials.
2. Prove that the function f is identically zero iff all the polynomials p_i are identically zero.
3. Find all solutions of each of the equations

$$y''(x) + 3y'(x) + 2y(x) = xe^{\lambda x}, \quad \lambda = -2, -1, 0, 1.$$

Hint: Use induction over n in part 2.

Q 11 ● [M] ↑ Q 10

Radioactive decay is known to be described by the differential equation $y'(t) + \alpha y(t) = 0$, where t denotes time and $y(t)$ is the amount of the original radioactive substance at time t. Hence $y(t) = y_0 \exp(-\alpha t)$. Prove that the time span in which the amount y_0 is halved depends neither on the initial amount nor on the starting time. Assuming as known the time span T_1 in which this amount is halved, find the time span in which this amount is reduced to $1/10$, $1/100$ of its initial value.

Q 12 • [M] ↑ Q 11

A sample is known to be a mixture of two radioactive substances with known half-lives h_1, h_2 respectively. Its amount has been measured at times t_1, t_2 and found to be a and b respectively. Find the original proportion of the substances. Solve the problem for the following data: $h_1 = 2$, $h_2 = 4$, $t_1 = 1$, $t_2 = 2$, $a = 11$, $b = 9$.

Examples Q 11 and Q 12 hint at the basic idea of the theory and use of radiocarbon dating. This is a reliable method to estimate the age of organic remnants from archaeological sites, developed by Willard Libby in 1949.

Q 13 ••

Some of the equations below for unknown functions f may have nontrivial polynomial solutions, i.e. by polynomials of degree > 0. Choose these equations and find their polynomial solution.

(a) $f(x) + f(y) = f(x + y)$,
(b) $f(x)f(y) = f(x + y)$,
(c) $f(x)f(y) = f(xy)$,
(d) $f(f(x)) = x + f(x)$.

Q 14 • ↓ Q 34 ↓ Q 35

The values $g_n(t)$ of any trigonometric polynomial of degree n can be expressed as the values of an (algebraic) polynomial of degree $2n$ on the unit circle, i.e. $g_n(t) = e^{-\mathbf{i}nt} G_{2n}(e^{\mathbf{i}t})$, where G_{2n} is a standard polynomial of degree $2n$.
Express the coefficients of G in terms of those in g.

Q 15 • ↑ Q 02

Find a cubic polynomial p such that $p(0) = p_0$, $p'(0) = m_0$, $p(1) = p_1$, $p'(1) = m_1$, where p_0, p_1, m_0, m_1 are given values.

6

Q 16 • ↑ Q 02 ↑ Q 15

For given values x_0, x_1 find a cubic polynomial p such that $p(x_0) = p_0$, $p'(x_0) = m_0$, $p(x_1) = p_1$, $p'(x_1) = m_1$, where p_0, p_1, m_0, m_1 are given values.

Q 17 • A 20 ↑ Q 04 ↑ Q 15

For two arcs given as $y = (x+1)^2$ for $x \le 0$ and $y = 1 + (x/2+1)^2$ for $x \ge 1$, respectively, find a path (a curve) connecting the two endpoints such that the resulting curve is an arc with a continuous derivative.

Polynomial interpolation schemes which satisfy requirements on functional values and also values of derivatives are known as Hermite interpolation.

Q 18 ••• ↓ PQ 18 ↑ Q 15 ↑ Q 16

Let the values x_k and y_k, $k = 0, 1, 2, \ldots, n$ be given. Find, if possible, a function $S(x)$ with the following properties:

1. $S(x_k) = y_k$ for $k = 0, 1, 2, \ldots, n$;
2. on each of the intervals (x_k, x_{k+1}) the function $S(x)$ is equal to a cubic polynomial;
3. $S''(x)$ is continuous.

Examples Q 15 – Q 18 are basic starting blocks for building a theory of cubic splines. Any of the functions $S(x)$ of Q 18 interpolates the given set of values. Various choices of initial or boundary values for M_k (see PQ 18) enable to further specify its properties.This theory and its various generalizations became an important tool in approximation theory. They are widely used in graphic packages in computer programs.

Q 19 •• [M] ↑ Q 02

Design a smooth widening profile of a cup with radii 3 cm at the bottom and 5 cm at the top and such that its content is 0.2 liters. Find the positions of the calibration marks for each 50 milliliters.

Q 20 •• ↑ Q 02

Let $P_n(x)$ be a given polynomial of degree n. Find a polynomial y which satisfies the equation

$$y'(x) + ay(x-1) = P_n(x), \quad a \in \mathbf{R}$$

for all values of x. Start with a decision on the degree of y. Is the solution unique?

Hint: Start with $P_n(x) = 1$, further consider e.g. $P_n(x) = 2x^3 + 3x + 1$. Continue with $P_n(x) = b_2x^2 + b_1x + b_0$. For the general case take $P_n(x) = a_0 + a_1x + \cdots + a_nx^n$ and find the a_i.

Q 21 • ↑ Q 02

Find the power series $\sum_{i=0}^{\infty} a_i x^i$ of the function y which satisfies the differential equation $y' + \alpha y = 0$, with $y(0) = 1$.
Find the power series solution of the equations $y'' + \alpha^2 y = 0$ and $y'' - \alpha^2 y = 0$ with $y(0) = 1$, $y'(0) = 0$ and with $y(0) = 0$, $y'(0) = 1$. Note that the solution of each of the differential equations is well known.

Q 22 •• ↑ Q 02 ↑ Q 21

Find a general solution of the differential equation $y''(x) + xy(x) = 0$ in form of a power series. Find its particular solution for $y(0) = 1$, $y'(0) = 0$.

Q 23 •• ↑ Q 02 ↑ Q 21

Find the power series $\sum_{i=0}^{\infty} a_i x^i$ of the function y which satisfies the differential equation

$$x^2 y''(x) + xy'(x) + x^2 y(x) = 0$$

and $y(0) = 1$, $y'(0) = 0$.
Find the radius of convergence of the obtained series and compare its sum with the power series of the Bessel function $J_0(x)$.

6

Q 24 ●●

Find a polynomial P_n of degree n with positive real coefficients and with $P(0) = 1$ such that the function

$$A(t) = \frac{1}{P(\mathbf{i}t)P(-\mathbf{i}t)}$$

has $2n - 1$ derivatives equal to zero at $t = 0$, and $A(1) = 1/2$.

Hint: Solve successively for $n = 1, 2, 3$.

These polynomials are called Butterworth polynomials. They have applications in network theory, where it is desirable to find an approximation of the function $f(t) = 1$ for $-1 < t < 1$ and zero otherwise, such that this approximation is decrasing for positive values of t. For $n > 3$ these polynomials are not unique and for their definition to be suitable in applications additional requirements have to be imposed. This requirement demands that $P_n(z) \neq 0$ for all $\mathrm{Re}\, z \geq 0$.
For $n = 4$ there are two solutions of the problem: $P_{41} = 1 + 2.61313z + 3.41421z^2 + 2.61313z^3 + z^4$ and $P_{42} = 1 + 1.08239z + 0.585786z^2 + 1.08239z^3 + z^4$. Here P_{42} has a zero point with a positive real part and therefore it is unacceptable in network theoretical applications.

Q 25 ●●

Denote the ratio of two polynomials P_n and Q_m of degree n and m, respectively by $R[n, m]$. For the function $\exp(x)$ find $R[0, 1]$, $R[1, 1]$ and $R[0, 2]$ such that the power series of the difference

$$f(x) = \exp(x) - R[i, k]$$

has zero derivatives at the point zero of order as high as possible.

These functions R are called the Padé approximants of $\exp(x)$.

Q 26 ● **C 01 V 09** ↓ **Q 27**

The straight lines $2x + y = 4$, $x - y = 1$, $y = 5$ form a triangle. Find the point C with minimal sum of squared distances from its sides and the point D with minimal sum of squared distances from its vertices.

Q 27 •• [M] ↑ Q 26

The system of equations $2x + y = 4$, $x - y = 1$, $y = 5$ has no solution. For arbitrary values of ξ and η define the errors in each of the equations as follows: $e_1 = 2\xi + \eta - 4$, $e_2 = \xi - \eta - 1$, $e_3 = \eta - 5$.

(a) Find the values ξ, η which minimize the sum of squares of e_i, $i = 1, 2, 3$.
(b) Consider the 3×2 matrix A of the system and compare your result with the results of the command `PseudoInverse [A]. (4,1,5)` in MATHEMATICA® or `LeastSquares(A,b)` of the LinearAlgebra Package in Maple®.
(c) Can you explain the difference between the solutions of Q 26 and Q 27 (a)?

Q 28 • [M] ↓ Q 29

(a) Let a process be measured at time instances $t_1 < t_2 < \cdots < t_n$ with results x_i, $i = 1, 2, \ldots, n$, respectively. Find a straight line $x = at + b$ such that the sum of squared distances of the n points (t_i, x_i) from this straight line is the smallest possible.
(b) Minimize the sum $\sum_{i=1}^{n}(x_i - at_i - b)^2$.
(c) Compare your result with that of the command $Fit[((t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n)), (1, t), t]$ in MATHEMATICA® or `LeastSquares(xydata,v)` in the CurveFitting Package of Maple®, where the symbols are replaced with actual data.

This procedure, called also linear fit, is the most common model of prediction of the future of some developing phenomena e.g. that of prices, production, temperature, concentration etc. It is also known as extrapolation. When little is known on the nature of the process such predictions have to be taken with care. They often yield wrong results, especially when applied to long-term predictions. The least squares method in this linear variant is also important in regression analysis of statistical data.

Q 29 •• ↑ Q 28

For given data as in Q 28 (a) find a polynomial P_m of degree $m < n$ minimizing the sum $\sum_{i=1}^{n}(x_i - P_m(t_i))^2$.
Does this polynomial minimize the sum of squared distances of the n points (t_i, x_i) from the graph of the polynomial P_m?

6

Q 30 • ↑ **Q 28**

A process is known to be described by a differential equation $y' = -ky$. Its output has been measured at the time instants $t = 0, 1, \dots, 10$ and the results are recorded with some measurement errors as follows (1.97, 1.90, 1.68, 1.52, 1.32, 1.16, 1.15, 0.97, 0.91, 0.83, 0.72). Find or estimate the initial value and the time constant k by the least squares method.

Q 31 ••

In mathematical biology the solution of the differential equation

$$y'(t) = ry(t)(1 - y(t)/b)$$

with $r > 0$, $b > 0$ and $y(0) > 0$ is often taken as the mathematical model of growth. Prove that $y(t) > b$ for $t > 0$. Find a method of estimating the parameters r, b and $y(0)$ from observed values of the output. Consider the following data of $(t, y(t))$: $((0, 2.28474), (1, 2.20772), (2, 2.17057), (3, 2.18135), (4, 2.11884), (5, 2.11421), (6, 2.11719), (7, 2.07284), (8, 2.07877), (9, 2.07582), (10, 2.04431))$.

Hint: Find the solution of the differential equation.

The equation is known as the logistic differential equation. Its numerous applications in biology, demography, economics, chemistry, mathematical psychology, probability, sociology and statistics led to mathematical investigations of its properties, including discrete versions and many of its generalizations.

Q 32 • **[M]**

The distance between two real functions f, g in the space $L_2(a, b)$, where (a, b) is a given interval, is defined by

$$d(f, g) = \left(\int_a^b (f(x) - g(x))^2 dx \right)^{1/2}.$$

Among all the functions $f(x) = kx + q$ find the one with the smallest distance from the function $\sin x$ on the interval $(-\pi/2, \pi/2)$.

Q 33 ●● ↑ **Q 32**

Let f be a function defined on $(0, 2\pi)$ and let

$$T_n(x) = \sum_{j=1}^{n} q_j \sin jx.$$

Find such coefficients q_j which give the minimal value to the integral

$$\int_0^{2\pi} (f(x) - T_n(x))^2.$$

Can this result be generalized?

Hint: Note the orthogonality of the system $\{\sin nx\}$ in the space $L_2(0, 2\pi)$.

Q 34 ●

Let T_1, T_2 be trigonometric polynomials of degree n and m, respectively. Find conditions under which $\int_0^t T_1(x)dx$ and T_1' are trigonometric polynomials and find their degrees.
Prove that $T_1 T_2$ is a trigonometric polynomial and find its degree.

Q 35 ● ↑ **Q 14**

A trigonometric polynomial of degree n cannot have more than $2n$ zeros in $[0, 2\pi)$. Find a proof.

Q 36 ●● ↑ **Q 34**

Prove that the function $h(t) = \prod_{k=1}^{2n} \sin \frac{t-t_k}{2}$ with $0 \le t_k < 2\pi$ is a trigonometric polynomial of degree n.

Hint: Find that $\prod_{k=1}^{2} \sin \frac{t-t_k}{2}$ can be written in the form $a_0 + a_1 \cos t + b_1 \sin t$. Then proceed by induction with the help of the last statement in Q 34.

6

Q 37 ●● ↑ Q 05 ↑ Q 36

Let $2n+1$ mutually different values t_k, $0 \le t_k < 2\pi$ be given. Find trigonometric polynomials g_k of period 2π and of degree $\le n$ such that $g_k(t_i) = \delta_{ik}$.

Hint: Take $\omega(t) = \prod_{k=0}^{2n} \sin \frac{t-t_k}{2}$ and proceed in a way similar to forming Lagrange polynomials in Q 05.

Supplementary Material

Definitions

A function f is said to interpolate the function g at a system of nodes x_i, $i = 1, 2, \ldots, n$, if $f(x_i) = g(x_i)$ for all $i = 1, 2, \ldots, n$.

A system of functions $u_i(x)$, $i = 1, 2, \ldots, m$, defined on the interval $[a, b]$ is called orthogonal on a system of nodes $x_k \in [a, b]$, $k = 1, 2, \ldots, n$ if

$$\sum_{i=1}^{m} u_i(x_k)u_i(x_q) = \delta_{kq} \quad \text{for all } k, q = 1, 2, , \ldots, n.$$

The function

$$g(t) = a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

is called a trigonometric polynomial of degree n and period 2π.

Let f be a function with r continuous derivatives at the point $x = 0$. The rational function $R[n, m] = \frac{p_n(x)}{q_m(x)}$ with polynomials p_n, q_m of degrees n and m, respectively, and with $n + m < r$, is called the Padé approximants of order (n, m) of the function $f(x)$ if all the derivatives of order $0, 1, 2, \ldots, n+m$ of their difference $f(x) - \frac{p_n(x)}{q_m(x)}$ vanish at $x = 0$.

In the set $L_2(a, b)$ of real measurable functions f, defined on the interval (a, b) for which the integral $\int_a^b |f(x)|^2 dx < \infty$ the real number

$$d(f, g) = \sqrt{\int_a^b |f(x) - g(x)|^2 dx}$$

is a distance from f to g.
The inner product $(f, g) = \int_a^b f(x)\overline{g}(x)dx$, where the bar denotes Conjugate.

The formula

$$x(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{\sin(t - nT)}{t - nT}$$

with $T < \frac{1}{2B}$ is called the Whittaker–Shannon interpolation formula, provided the Fourier integral $X(\omega)$ of the function x is vanishing for $|\omega| > B$. Such functions x are called band-limited.

For an arbitrary $n \times m$ matrix A the matrix A^+ is called the pseudoinverse (also the Moore–Penrose inverse) of A iff

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^* = AA^+$
4. $(A^+A)^* = A^+A$

Here M^* is the conjugate transpose of M.

Theorems

A

For any matrix A there exists a unique pseudoinverse matrix A^+.

6

Plans of Solution

PQ 18

1. Assume that $x_k < x_{k+1}$ and denote $h_k = x_{k+1} - x_k$, $S''(x_k) = M_k$. On each of the subintervals $[x_k, x_{k+1}]$ the second derivative of S, say S''_k, is a linear polynomial. Find that S''_k, can be written as follows

$$S''_k(x) = M_k \frac{x_{k+1} - x}{h_k} + M_{k+1} \frac{x - x_k}{h_k}, \quad x \in [x_k, x_{k+1}].$$

2. By integration obtain successively $S'_k(x)$ and $S_k(x)$.

$$S'_k(x) = -M_k \frac{(x_{k+1} - x)^2}{2h_k} + M_{k+1} \frac{(x - x_k)^2}{2h_k} + C_k,$$

$$S_k(x) = M_k \frac{(x_{k+1} - x)^3}{6h_k} + M_{k+1} \frac{(x - x_k)^3}{6h_k} + C_k(x - x_k) + D_k.$$

3. Calculate the values of the integration constants C_k and D_k from the conditions

$$y_{k+1} = S_k(x_{k+1}) \quad \text{and} \quad y_k = S_k(x_k)$$

   to obtain $C_k = \frac{y_{k+1} - y_k}{h_k} + \frac{M_k - M_{k+1}}{6} h_k$ and $D_k = y_k - \frac{h_k^2}{6} M_k$.
4. The condition $S'(x_k) = S'_{k-1}(x_k)$ finally gives the second order linear difference equation

$$\frac{h_{k-1}}{6} M_{k-1} + \frac{h_k + h_{k-1}}{3} M_k + \frac{h_k}{6} M_{k+1} = \frac{y_k - y_{k-1}}{h_{k-1}} - \frac{y_{k+1} - y_k}{h_k}$$

   for the unknown values M_k of the second derivatives. It has a unique solution with two additional conditions given. We may state that with given data there exist infinitely many functions $S(x)$ satisfying the requirements.

Further References

Spline functions are dealt with in many monographs and textbooks, e.g.

Ahlberg J.H., Nilson E.N., Walsh J.L., The Theory of Splines and their Application, Academic Press, New York, 1967.

Maximal and Minimal Values 7

Many mathematical models of physical or biological processes are created so as to find or verify conjectures on extreme values of the involved variables. This is extremely important when safety margins have to be set and/or parameters of a design have to be found which enable proper functioning of a device under extreme conditions. Mathematical methods often make it possible to avoid expensive experiments or simulations.

In this chapter we will learn when and how the well known fact on local extremes of differentiable functions can be applied, i.e. the fact that for such functions local extremes can appear only at points where the first derivatives vanish. Solution of the corresponding equations is often rather difficult. In some problems these equations are not directly accessible and other considerations have to be applied. The solution of problems marked as [M] and many others can be made easier when using MATHEMATICA®, although only very few of them can be solved directly. The user is strongly advised to learn the syntax and semantics of some of the commands. In this chapter the following commands (and related ones) may help:

In MATHEMATICA®:

```
D, Solve, NSolve, FindRoot, FindMinimum,
FindMaximum
```

In Maple®:

```
max, diff, solve, fsolve, Optimization[Minimize],
Optimization[Maximize]
```

In most of the examples numerical experiments may give a starting point for reasoning or help to verify initial conjectures. Explanation of mathematical terms and concepts can also be found on the Internet, e.g. at www.mathworld.wolfram.com or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://eom.springer.de.

J. Gregor, J. Tišer, *Discovering Mathematics*,
DOI 10.1007/978-0-85729-064-9_8,

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Suggestions

- Easy examples of application of standard methods are V 01, V 02, V 03, V 04, V 11, V 14, V 21.
- Constrained extremes are illustrated by V 15, V 18, V 19, V 23.

Problems

V 01 • ↓ **V 02**

Let f be an even concave positive function defined on the interval $[-1, 1]$ such that $f(-1) = f(1) = 0$. Consider the region between the x axis and the graph of the function f. Find the rectangle with its sides parallel to the coordinate axes of maximal area inscribed in this region.

Hint: Use Newton's formula to find an approximation of the solution.

V 02 •• **[M]** ↑ **V 01**

Find the rectangle of maximal area with sides parallel to the coordinate axes inscribed in the region bounded by $[-1, 1]$ and the graph of the function $f(x) = (1 - x^2)(1 + x)$.

Hint: Find conditions for a, b such that $f(a) = f(b)$.

V 03 • ↓ **V 04**

A rod of given length d has to be transported in horizontal position through an L-shaped corridor with a given width a of one of its parts, $a < d$. Find the minimal width of the other part of the corridor. Consider e.g. $d = 5$, $a = 1$.

V 04 • [M] ↑ V 03

A truck of width w (including safety margin) has to pass an L-shaped crossing. The widths of the roads are a and b, respectively. Find the maximal length of the truck. Consider e.g. $a = 6$, $b = 5$, $w = 2$.

V 05 •

Consider a straight line p in the plane and points A, B in the same half-plane. Find a point C on p such that the sum of distances from A to C and from C to B is minimal. Find its geometrical construction.

V 06 • ↑ V 05

Considering a plane ρ in three-dimensional space, formulate and solve the problem analogous to V 05. In this case the three points A, B, C determine a plane. Characterize its position with respect to the plane ρ.

V 07 •• ↑ V 05

Consider the ellipse $2x^2 + y^2 = 2$ and a pair of points: $\{A, B\} = \{(3, 2), (-1, 4)\}$. Find a point G on the ellipse such that the sum of distances $d(A, G) + d(G, B)$ is minimal. (Note that both A and B are outside the ellipse.) In general, if the point A is fixed and both segments $\overline{AG}$, $\overline{GB}$ have to be in the outer region of the ellipse then this problem has a solution only if B satisfies some additional conditions. Find these conditions for the given point A.

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V 08 •• **PV 08** ↑ **V 05** ↑ **V 06**

Consider a straight line p in $\mathbf{R}^2$ and points A, B in opposite half-planes. A particle is moving from A to B on straight lines with velocity v_1, v_2, respectively. Find the condition imposed upon the point C on the straight line p such that the particle reaches the point B in minimal time. Calculate the point C if p is the x-axis, $A = (0,3), = (4,-1), v_1 = 1, v_2 = 3$.

The reader may recognize that examples V 05–V 08 are based on Fermat's principle or the principle of least time, which is the idea that the path taken between two points by a ray of light is the path that can be traversed in the least time. While V 08 expresses Snell's law of refraction, the others are based on the law of reflexion.

V 09 ••• **PV 09** ↑ **V 05** ↑ **V 06**

For a triangle with vertices at $(-1,0)$, $(1,0)$, (p,q), $q > 0$ find a point F such that the sum of its distances from the vertices of the triangle is minimal.

The point F in this problem can be identified with the Fermat point of the triangle. It first appeared in a letter of Fermat to Torricelli. Its construction was published by Viviani in 1659. A number of its properties can be found e.g. at http://mathworld.wolfram.com/FirstFermatPoint.html and many other places.

V 10 ••

A dumped harmonic oscillator with the dumping force proportional to the square of velocity can be described by the differential equation

$$y'' + a(y')^2 + by = 0.$$

Motion of this type occurs in fluids with moderate values of velocity. With given initial value $y(0) < 1/(2a)$ and $y'(0) = 0$ find the displacement y at which the motion attains its maximum velocity for $a = 0.01$, $b = 0.5$ and $y(0) = -100$.

Hint: Introduce the function $v(y) = y'^2$ and solve the resulting linear differential equation for the unknown function v.

V 11 • **↑ V 05 ↑ V 06**

Two trains are moving at constant speed v_1 and v_2 on straight lines towards their crosspoint, starting at points A and B, respectively. Find the smallest distance between them.

Hint: Choose a suitable coordinate system.

V 12 •• **PV 12**

In the n-dimensional space $\mathbf{R}^n$ consider two straight lines with directional vectors a, α, respectively, passing through the points b, β, respectively. Find the distance between them.

V 13 ••• **A 05** **↑ V 05 ↑ V 09 ↑ V 11**

A point is moving from point $A = (0, a)$ to point $B = (b, 0)$, $a, b > 0$ across n horizontal strips of width $d = a/n$ with velocity $v_k = v\sqrt{k}$, $k = 1, 2, \ldots, n$ in the k-th strip (the strips are numbered from above). In each of the strips the point moves along a straight line. Find the path along which the point comes to the point B in the shortest possible time. Verify your results for $n = 4$, $v = 1$, $b = 10$, $d = 2$.

Hint: Consider as independent variables the (acute) angles of the path with the verticals. Compare the graph of the resulting path for fixed values a, b and for $v = 1$ and with increasing value of n with the curve

$$x(t) = \frac{b}{\pi/2 - 1}(t - \sin t), \qquad y(t) = a - a(1 - \cos t), \qquad 0 \le t \le \pi/2,$$

which is a cycloid.

This problem and its results are closely connected to the important brachystochrone problem, the solution of which was given at the end of the 17th century by Newton, Leibnitz, L'Hospital, Johann and Jacob Bernoulli. It can be formulated as follows: Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time. It has been found that the curve is an arc of a cycloid. These investigations led to development of an important part of mathematics known as the calculus of variations.

7

V 14 ●● ↓ **V 15**

Find the distance between the ellipse $x^2 + 2y^2 = 1$ and the point $(2, 3)$. Compare this distance with the length of the segment of the normal from the point to the ellipse.

V 15 ● ↑ **V 14**

A set of points is given by the following conditions:

$$\text{either} \quad -1 \leq x - y \leq 1, \quad \text{or} \quad 9x^2 + 4y^2 \leq 36.$$

Find the distance of the point $(1, 4)$ from this set.

Hint: For the set $A = B \cup C$ there is $d(x, A) = \min(d(x, B), d(x, C))$.

V 16 ● ↑ **V 05**

A billiard table has edges parallel to the coordinate axes and two opposite corners are at the origin and at the point $(10, 15)$. Two billiard balls are at the locations $(1, 5)$, $(4, 2)$. How can one ball hit the other ball if travelling directly in a straight line is not allowed? How many solutions can be found if at most one reflection point is allowed?

V 17 ●● ↑ **V 05** ↑ **V 16**

Solve V 16 again with two reflection points. How many solutions exist?

V 18 ●●

Given four points (x_i, y_i), $i = 1, 2, 3, 4$ of a rectangle, find the circumscribed ellipse of minimal area. Calculate the equation of such an ellipse for the points $(-2, 5)$, $(2, -5)$, $(2, 5)$, $(-2, -5)$.

Hint: Take into account the involved symmetries.

V 19 •• **X 15** ↑ **V 18**

For the ellipse $x^2 + xy + 2y^2 = 3$ find the half-axes and the coordinates of the foci. Design a procedure solving this problem for the ellipse given as

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_{33} = 0$$

with $a_{11}a_{22} - a_{12}^2 > 0$.

Hint: Find the maximal and minimal square of the distance of (x, y) from the origin, subject to the constraint $x^2 + xy + 2y^2 = 3$.

V 20 ••

Let the function f have its local maximum at the point x_0. Prove or disprove the following statement: There exists a neighborhood U of x_0, such that in U the function f is increasing on the left and decreasing on the right of x_0.

Hint: Consider the function $f(x) = 2 - x^2(2 + \sin\frac{1}{x})$ for $x \neq 0$ and $f(0) = 2$.

V 21 • ↑ **V 05**

Find the dimensions of a can which has minimal surface area and a given volume V.

V 22 •• **E 05 E 09**

In the plane define a function

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

and prove that d is a distance function, i.e. it defines a metric in the plane. Find all points which are at a given distance d_0 from the origin.

7

V 23 ●● ↑ **V 19**

Given four points $(0,0)$, $(-2,3)$, $(4,1)$, $(2,4)$ of a parallelogram, find the circumscribed ellipse of minimal area.

Hint: Transpose the center of the parallelogram into the origin. The area of an ellipse with half-axes m and M is $\pi m M$. Find that for an ellipse with its equation in the form $ax^2 + 2bxy + cy^2 = f$, $f > 0$, its area A can be expressed as

$$A = \frac{\pi f}{\sqrt{ac - b^2}}.$$

V 24 ●●

Find the normalized quadratic polynomial $P_2(x)$ such that the value $\max_{x\in[-1,1]} |P_2(x)|$ is minimal.

V 25 ●●● ↑ **V 24**

Consider a normalized cubic polynomial $P(x)$ and denote by $x_{\min}$ and $x_{\max}$ the points of its local minimum and maximum, respectively.

1. Assuming that both $x_{\min}$ and $x_{\max}$ are in $(-1,1)$, find $P(x)$ such that
$$|P(1)| = |P(-1)| = |P(x_{\min})| = |P(x_{\max})|.$$
2. Assume that $|P(x_{\min})| = |P(x_{\max})|$ in the open interval $(-1,1)$ and find $P(x)$.

Hint: For $P(x) = ax^3 + bx^2 + cx + d$ the condition $|P(1)| = |P(-1)|$ implies $(c+1)(b+d) = 0$. Treat the other conditions similarly.

V 26 •• Q 07

Consider the function $f(x) = x^2 + x^3$ on the interval $[-1, 1]$. Find a quadratic polynomial $P(x) = ax^2 + bx + c$ such that

1. P interpolates the function at equidistant nodes;
2. P interpolates the function f at Chebyshev nodes, i.e. at $x = 0$, $x = \pm\frac{\sqrt{3}}{2}$;
3. P minimizes the value of $\int_{-1}^{1}(x^3 + x^2 - P(x))^2 dx$;
4. P minimizes the value of
$$\int_{-1}^{1}(x^3 + x^2 - P(x))^2\sqrt{1 + x^2}dx.$$
5. Draw graphs of your results.

V 27 ••

Let S be the set of functions

$$S = \{f_n : f_n(x) = |x + 2n|,\ n \in \mathbf{Z}\}.$$

Define $F(x) = \min_{f\in S} f(x)$ and find all local minima and maxima of F.

V 28 •• E 05

Let f be a twice-differentiable function for all $x \in \mathbf{R}$ and let

$$M_k = \sup_{x\in\mathbf{R}} |f^{(k)}(x)|, \quad k = 0, 1, 2.$$

Prove that $M_1^2 \le 2M_0M_2$.

Hint: Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}(f(x + 2h) - f(x)) - hf''(\xi).$$

7

V 29 •

A sequence is defined by the difference equation

$$y(n+2) - 2y(n+1) + y(n) = \frac{n^3}{16} - n, \qquad y(0) = y(1) = 1.$$

Find its term of minimal value.

Supplementary Material

Definitions

A pair (X, d) is called a metric space if X is a nonempty set and with any two points p, q of X there is associated a number $d(p, q)$, called the distance from p to q, such that

1. $d(p, q) > 0$ iff $p \neq q$, $d(p, p) = 0$;
2. $d(p, q) = d(q, p)$;
3. $d(p, q) \leq d(p, r) + d(r, q)$ for any p, q and $r \in X$.

For two sets A, B in a metric space their distance d is defined by

$$d(A, B) = \inf_{a \in A, b \in B} d(a, b).$$

Notice that d does not define a metric on the space of subsets of (X, d).

A polynomial is called normalized if the coefficient with its highest power equals 1.

Theorems

A

Suppose that the function f has a zero at α, i.e. $f(\alpha) = 0$. If f is continuously differentiable and its derivative is nonzero at α then there exists a neighborhood of α such that for all starting values x_0 in that neighborhood, the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

will converge to α.

B

Let $f, g : \mathbf{R}^2 \to \mathbf{R}$ be continuously differentiable. If f has a maximum on the set $\{(x, y) : g(x, y) = c\}$ at the point (x_0, y_0) then the function $L(x, y, \lambda) = f(x, y) + \lambda(g(x, y) - c)$ has a stationary point at (x_0, y_0, λ_0).

Plans of Solution

PV 08

1. Let p be the x-axis and put $A = (0, y_1)$, $B = (x_2, y_2)$, $y_1 y_2 < 0$, $C = (c, 0)$. The overall time will be

$$T(x) = \frac{\sqrt{y_1^2 + x^2}}{v_1} + \frac{\sqrt{(x - x_2)^2 + y_2^2}}{v_2}.$$

$T'(x) = 0$ gives

$$\frac{x}{v_1\sqrt{y_1^2 + x^2}} = \frac{x_2 - x}{v_2\sqrt{(x - x_2)^2 + y_2^2}}.$$

In general this equation is cumbersome to solve.

2. Introducing the angle α_1 of incidence of the beam from A to C and angle α_2 of reflection of this beam, notice that

$$\sin\alpha_1 = \frac{x}{\sqrt{y_1^2 + x^2}}, \qquad \sin\alpha_2 = \frac{x_2 - x}{\sqrt{(x - x_2)^2 + y_2^2}}.$$

3. Deduce from $T'(x) = 0$ the so-called Snell's law of refraction

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{v_1}{v_2}.$$

4. The x coordinate of point C is the solution of equation

$$\frac{v_2 x}{\sqrt{y_1^2 + x^2}} = \frac{v_1(x_2 - x)}{\sqrt{(x - x_2)^2 + y_2^2}}.$$

7

PV 09

1. Verify that the direct way to minimize the sum S of distances is unsuitable.
2. Denote by S_1, S_2, S_3 the distances of the point F to each of the vertices. To minimize $S = S_1 + S_2 + S_3$ consider that

   $$\operatorname{grad} S = \operatorname{grad} S_1 + \operatorname{grad} S_2 + \operatorname{grad} S_3 = 0.$$

   Observe that $\operatorname{grad} S_i$ is a unit vector pointing from F to the corresponding vertex. Hence the segments joining F with the vertices divide the angle 2π into three equal parts.
3. The point F lies on each of the three circles with the common property: Each of these circles is determined by two vertices of the triangle and a third point which forms a regular triangle with them.
4. Find that for any two given points, say (m, n), (r, s) a third vertex of a regular triangle is at

   $$\left(\frac{m+r}{2} \pm \frac{\sqrt{3}(n-s)}{2}, \ \frac{n+s}{2} \pm \frac{\sqrt{3}(r-m)}{2}\right)$$

   and for the pairs of points $(-1, 0)$, $(1, 0)$ and $(1, 0)$, (p, q) respectively, choose the proper sign. (In both cases the $+$ sign.)
5. Find the equations of the two circles. One of their common points is the point F.

PV 12

1. The two straight lines can be described as $X = at + b$, $Y = \alpha s + \beta$, where X, Y, a, α, b, β are vectors. Denoting by $\langle \cdot, \cdot \rangle$ the scalar product, for the distance d of any two points of the straight lines obtain

   $$d^2(X, Y) = \langle X - Y, X - Y \rangle.$$

2. Differentiate this equation with respect to t and s and equating these derivatives to zero obtain the system of equations

   $$|a|^2 t - \langle a, \alpha \rangle s = \langle a, \beta - b \rangle,$$
   $$-\langle \alpha, \alpha \rangle t + |\alpha|^2 s = -\langle \alpha, \beta - b \rangle.$$

3. The values t and s can be calculated iff $|a|^2 |\alpha|^2 - \langle a, \alpha \rangle^2 \neq 0$, which excludes parallel straight lines from further considerations.
4. From the system in step 2 obtain the identity

   $$|a|^2 t^2 + |\alpha|^2 s^2 - 2ts \langle a, \alpha \rangle = \langle at - \alpha s, \beta - b \rangle$$

   and finally $d^2 = \langle at - \alpha s, b - \beta \rangle + |b - \beta|^2$, where t and s solve the system of equations in step 2.

Part III
Applications

Arcs and Curves 8

Planar curves are one of the oldest topics of geometry. Basic results were known by Greek mathematicians several centuries BC. Apollonius of Perga (around 200 BC) introduced the names of ellipse, hyperbola and parabola and described many of their properties as conic sections. Many results were obtained later by algebraic methods (i.e. with the introduction of coordinate systems) and using methods of calculus; some problems remained open till the end of 19th century, first of all the exact concept of a curve. While Euclid spoke of curves as objects which have length but no width, the concept of length turned out to be of basic importance not only for geometry but also for physical concepts of forces, velocities and momentums.

In this chapter you will learn various ways how to describe and define curves. Methods of studying their shape, length and other properties will be presented. For the sake of simplicity mostly planar curves will be discussed.
The solution of problems marked [M] and many others can be made easier when using software packages comparable to S. Wolfram's MATHEMATICA®, although only very few of them can be solved directly. The user is strongly advised to learn the syntax and semantics of some of the commands. In this chapter the following commands (and related ones) may help:
In MATHEMATICA®:

```
Plot, Plot3D, ParametricPlot, ParametricPlot3D,
ImplicitPlot, PolarPlot, ContourPlot
```

and the package

```
Graphics`ComplexMap`.
```

In Maple®:

```
plot, plot3d, plots[conformal], plots[animate],
pointplot, contourplot
```

J. Gregor, J. Tišer, *Discovering Mathematics*,
DOI 10.1007/978-0-85729-064-9_9,

8

The

```
Interactive Plot Builder
```

contains many useful options.

In most of the examples numerical experiments may give a starting point of reasoning or verify initial conjectures. Explanation of mathematical terms and concepts can be found on the Internet, e.g. at www.mathworld.wolfram.com or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://eom.springer.de.

Suggestions

- A good start to deal with various ways of describing of a curve is in A 01, A 02, A 03, A 04, A 09.
- A curve and its length must be carefully defined as shown in A 07, A 12, A 29, A 32.
- Many important properties of curves and sources of their application are connected with tangents of curves. This concept is dealt with in A 08, A 10, A 11, A 14 – A 28 and some of these problems have many variants and interesting modifications.
- Problems of dynamic of motion often demand description of curves by differential equations. Examples A 37 – A 42 illustrate such situations.
- Curves in R^3 are the subject of A 18, A 19, A 23.

All readers are strongly encouraged to modify, generalize or simplify the formulated problems, to find alternative formulations and formulate and solve their own examples and find the context of these problems to the given ones.

Problems

A 01 • ↓ **A 02** ↓ **A 03** ↓ **A 04** ↓ **A 09**

Each of the following equations defines a set of points A_i in the (x, y) plane. What have these sets in common and how would you describe differences between them?

$$A_1 = \{(x, y) : x^2 + y^2 = 1\},$$
$$A_2 = \{(x, y) : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi\},$$
$$A_3 = \{(x, y) : z = e^{\mathbf{i}n},\ z = x + \mathbf{i}y,\ n \in \mathbf{Z}\}, \quad \mathbf{i}^2 = -1,$$
$$A_4 = \{(x, y) : x = \cos e^t,\ y = \sin e^t,\ t \ge 0\},$$
$$A_5 = \{(x, y) : x^6 + 3x^2y^2(x^2 + y^2) + y^6 = 1\},$$
$$A_6 = \{(x, y) : x^2 - 2x + y^2 = 0\},$$
$$A_7 = \left\{(x, y) : \left|\frac{z-a}{z+a}\right| = k,\ a = \frac{1}{2}\left|k - \frac{1}{k}\right|,\ a, k \text{ real},\ z = x + \mathbf{i}y\right\}$$

A 02 • ↑ **A 01** ↓ **A 08**

Show that the arc defined by

$$x(t) = a_{11}t + a_{22}, \qquad y(t) = b_{11}t + b_{22}, \quad t \in \mathbf{I} \subset \mathbf{R_1},$$

with at least one of the values $a_{11}b_{11} \neq 0$ is a segment of a straight line. Express in terms of a_{ik}, b_{ik} the equation of this line in the form $ax + by + c = 0$.

Hint: Eliminate the variable t from the two equations.

A 03 • ↑ **A 01** ↑ **A 02** ↓ **A 21**

Can you prove that the arc defined by

$$x(t) = a_{11}t^2 + a_{12}t + a_{13},$$
$$y(t) = b_{11}t^2 + b_{12}t + b_{13},$$
$$t \in \mathbf{I} \subset \mathbf{R_1}, \quad a_{11}b_{12} - a_{12}b_{11} \neq 0$$

is always an arc of a parabola?

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Hint: Without loss of generality we may assume $a_{13} = b_{13} = 0$. Why?

A 04 • **[M]** ↓ **A 25**

Consider the equations $x^n + y^n - nxy = 0$. Find the set of points satisfying these equations for $n = 1, 2, 3, 4$. Guess the shape of these sets for increasing even and odd values of n, respectively. Find the symmetries of the curve.

Hint: The package `Graphics'ImplicitPlot'` in MATHEMATICA® or the `Interactive Plot Builder` in Maple® could be useful.

The curve with $n = 3$ in this example is called the folium of Descartes.

A 05 ••

Find conditions in terms of r and v for the curve

$$x = rt - v\sin t, \qquad y = r - v\cos t$$

to have no self-crossings.

These curves are known as cycloids. They are formed by a point fixed on, fixed inside or fixed outside a circle rolling along a straight line. You may try to derive the equation from these data denoting by r the radius of the circle and by v the distance of the fixed point from its center.

A 06 • **[M]** ↑ **A 01** ↑ **A 02**

How can you decide whether two arcs in $\mathbf{R}^2$ given by their parametric equations have a common point? Examine the common points of a beam $x = 2t + 1$, $y = -t + 1$, $t \geq 0$ and a circle

$$x = 3\cos t, \qquad y = 3\sin t, \qquad 0 \leq t \leq 2\pi$$

in $\mathbf{R}^2$.

Hint: Take into account that the variable t has a different meaning in the two parametric descriptions.

A 07 •• [M] **P 04**

The curve defined as

$$x = a\sin(\omega_1 t), \qquad y = b\sin(\omega_2 t + \phi), \quad 0 \le t \le T$$

is called Lissajou's curve. It describes a superposition of two orthogonal harmonic oscillations.

1. Find sufficient conditions on its parameters ensuring that the curve is closed.
2. Find conditions for the curve passing only points of a segment of a straight line and find this segment.
3. Characterize the behavior of the curve when it is not closed and T is large.

Hint: Prove that an arc $x = x(t)$, $y = y(t)$ is closed if both functions x, y are periodic with the same period.

A 08 • ↓ **A 12** ↓ **A 14** ↓ **A 15** ↓ **A 16** ↓ **A 20** ↓ **A 25** ↓ **A 26** ↓ **A 28**

Construct a tangent line to the graph of the function $y = f(x)$ at the point $x = x_0$ assuming that $f'(x)$ exists.
Construct a tangent line to the arc $x = x(t)$, $y = y(t)$ at a point $(x(t_0), y(t_0))$ and a tangent line to the arc $x = x(t)$, $y = y(t)$, $z = z(t)$ at a point $(x(t_0), y(t_0), z(t_0))$.
Give the meaning (definition) of the concept of half-tangent.

A 09 •• [M] ↑ **A 01**

An alternative way to determine a planar curve is in polar coordinates $x = r\cos t$, $y = r\sin t$.

1. Describe the curves $r = \sin(nt)$, where (r, t) are polar coordinates, for various, e.g. integer, values of n.
2. Show that the equation

   $$r = \frac{p}{1 + \epsilon\cos t}$$

   for any $p > 0$, $\epsilon \ge 0$ represents a conic. Find its position in the Cartesian coordinate system and the meaning of the constant ϵ for the type of conic.

Hint: Note that it must be $r \geq 0$. Use the package `"PolarPlot"` and `Interactive Plot Builder` in MATHEMATICA® and Maple®, respectively.

A 10 ●● ↑ **A 08** ↓ **A 11**

Find the part of the ellipse $x^2 + 2y^2 = 4$ which is visible from the point $(3, 3)$.

Hint: First prove that here the two common points of the straight line $b^2 x x_1 + a^2 y y_1 = a^2 b^2$ and the ellipse $b^2 x^2 + a^2 y^2 = a^2 b^2$ both lie on tangents to the ellipse passing through the point (x_1, y_1).

A 11 ●●● ↑ **A 08** ↑ **A 10**

1. How many observation points at a fixed distance from the center of a unit circle are necessary to monitor any point on its circumference?
2. How many observation points at a fixed distance from the center of a unit sphere are necessary to monitor any point on its surface?

A 12 ●● ↑ **A 08**

Describe a method to investigate the behavior of a curve $x = x(t)$, $y = y(t)$ in a neighborhood of a value of the parameter t for which both the derivatives of x and y vanish or do not exist. Examine the following arcs:

1. $x(t) = at^n$, $y(t) = bt^m$, $-1 \leq t \leq 1$;
2. $x(t) = at^n P_k(t)$, $y(t) = bt^n Q_j(t)$, $-1 \leq t \leq 1$, where P, Q are polynomials of degree k, j, respectively, and $P_k(0)Q_j(0) \neq 0$;
3. $x(t) = t + 5|t|$, $y(t) = -t + 7|t^3|$, $-1 \leq t \leq 1$.

Hint: Use the concept of a half-tangent. Consider the parity of n and m. Calculate, if they exist, the limits of $\dot{y}/\dot{x}$ for nonnegative and nonpositive values of t. Find the meaning of this ratio.

A 13 ●● ↑ **A 01** ↑ **A 03** ↑ **A 05**

Considering t to be time, the parametric equations of a curve can be viewed as describing motion along the curve. Find the vector of velocity. Explain in these terms the assumption that $x'(t)$ and $y'(t)$ are both nonzero, which is considered in the definition of an arc.
Analyze the motion described by:

1. $x = at^n + b$, $y = ct^n + d$, $0 \le t \le 1$ for various positive integer values of n;
2. $x = a \sin t + b$, $y = c \sin t + d$, $t \in [0, 2\pi)$.

A 14 ● ↑ **A 08**

The angle of two curves is defined as the angle of their tangents at the crossing point. Determine the angle of two parabolas $y = ax^2$, $x = ay^2$, $a > 0$.

A 15 ● ↑ **A 08** ↑ **A 14**

Prove that any of the parabolas $y = ax^2$ crosses any of the ellipses $x^2 + 2y^2 = b^2$, $a, b \in \mathbf{R}$ at a right angle.

Hint: Find the product of the slopes of tangents to each of the curves.

A 16 ● ↑ **A 08** ↑ **A 15**

Two families of curves F, G are called orthogonal trajectories if any curve of F crosses any curve of G at a right angle. Find the orthogonal trajectory of a family on concentric circles with the center at the origin.

Orthogonal trajectories may give rise to suitable coordinate systems. The one considered here is the system of polar coordinates. One of the important applications of orthogonal trajectories is in field theory.

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A 17 • **[M]** **M 44 ↑ A 14 ↑ A 15**

Let $f : \mathbf{C} \to \mathbf{C}$ be a holomorphic function of a complex variable z. Prove that the curves

$$\operatorname{Re} f(z) = k \quad \text{and} \quad \operatorname{Im} f(z) = q, \quad k, q \in \mathbf{R},$$

form an orthogonal trajectory.

Hint: Recall the Cauchy Riemann equations.

The package `Graphics'ComplexMap'` or the command `ParametricPlot` in MATHEMATICA® and the command `plots[conformal]` in Maple® yields a tool for experimenting with various orthogonal trajectories. Planar pictures of electric and/or magnetic fields of various origin can be handled by these packages. Experimenting with these tools gives interesting results.

A 18 • **[M]** **M 19 ↑ A 14 ↓ A 19**

A curve is given by

$$x = \alpha t \cos t, \qquad y = \alpha t \sin t, \qquad z = 2 - \alpha t, \quad 0 \le t \le 2/\alpha.$$

Describe its behavior and find the significance of the parameter α.

Hint: Use the command `ParametricPlot3D` in MATHEMATICA® or `plot` in Maple®.

A 19 •• **↓ PA 19 ↑ A 18**

On the globe (i.e. at a negligible altitude) an airplane moves at a constant speed v on a path which crosses all lines of latitude at a constant angle $\alpha \in [0, \pi/2)$. Find the equation of the path and the position of the plane at time t when the starting position is given.
Find the time needed to reach the pole starting in time $t = 0$ at latitude $\theta = 0$.

The curve is called a loxodromic curve. Its name is due to V. Snellius (1624).

A 20 • **[M]** **Q 17 ↑ A 08↓ A 21**

In design the following problem may arise: Given two planar arcs, join two of their endpoints by an arc in such a way that the resulting shape is as smooth (as nice) as possible. Solve this problem for the negative half-axis x and for the arc $x = 1 + 2t$, $y = 2 + 3t$, $t \geq 0$.

Hint: Consider polynomials x, y of a real variable t and of suitable degree; find its coefficients.

The direct polynomial solution, although possible, is not always a nice solution and it is not very flexible. A better solution is given by so-called Bezier curves.

A 21 •• **[M]** **↑ A 20↓ A 22**

Bezier curves are defined by their parametric equations $x = x(t)$, $y = y(t)$, $0 \leq t \leq 1$, where x, y are cubic polynomials. Their coefficients are determined by four points (x_i, y_i), $i = 0, 1, 2, 3$ as follows: The endpoints of the curve are (x_0, y_0) and (x_3, y_3). The tangent of the curve at (x_0, y_0) is directed towards (x_1, y_1) and the tangent of the curve at (x_3, y_3) is directed towards (x_2, y_2).
Express the coefficients of x and y in terms of the four basic points. Are these coefficients uniquely determined? After some experiments describe why the 'middle points' are called control points.

Bezier curves were originally designed as a design tool for creating shapes of desired properties. One of their applications is in creating graphics described in the PostScript language.

A 22 •• **[M]** **↑ A 21**

Design the shape of a bowl with the diameters d and u at the bottom and top given, with an upper bound of its height and with a given volume V.
Find also the positions of calibration marks for $V/4$, $V/2$, $3V/4$, V.
Start with $d = 5$, $u = 7$, $h = 20$ (cm), $V = 0.5$ liters.

A 23 • **[M]** ↑ **A 21**

Join by a Bezier curve two points $A, B \in R^3$ (e.g. design a road in a mountainous landscape). Use the following data: $A = (0, 0, 0)$, $B = (2, 3, 3)$; avoid as much as possible the segments $s_1 = \overline{(0, 1, z), (1, 1, z)}$, $s_2 = \overline{(1, 2, z), (2, 2, z)}$, for a fixed $z > 0$.

Hint: The Bezier curve in $\mathbf{R}^3$ is again given by (here 3) parametric equations constructed as in A 21.

A 24 •• ↓ **A 35** ↓ **A 36**

Approximate the area of a planar star-shaped domain when coordinates of a sufficient number of points on its border are given.

Hint: The area of a triangle can be expressed as $P = 1/2ab \sin \gamma$, where the two sides of length a, b, respectively form the angle γ.

A 25 • **[M]** ↑ **A 04** ↑ **A 08** ↓ **A 26**

Find all tangent lines of the curve defined by the equation $x^3 + y^3 - 3xy = 0$ at the points with a given x coordinate, $x \neq 0$.

A 26 • ↓ **A 27** ↓ **A 28**

An arc uniquely defines a (one parametric) set of straight lines, its tangent lines. Can you find such a family $y = k(\lambda)x + q(\lambda)$ of tangents to the curve $x = t$, $y = \sin t$, $0 \leq t \leq 2\pi$?

A 27 • ↑ **A 26**

Find the curves for which the family of tangent lines is the following:

(a) $y = 2\lambda x - \lambda^2$,
(b) $y = x \sin \nu + \cos \nu$, $\nu \in (-\pi/2, \pi/2)$,
(c) $y = -x \cot t + 1/\sin t$, $t \in (0, \pi)$.

A 28 •• ↓ **PA 28** ↑ **A 26**

Can a curve be defined by a one-parametric set of its tangents? Find the curve when its tangent lines are given as $y = k(\lambda)x + q(\lambda)$, where k and q are nonconstant differentiable functions of the real variable λ.
Consider e.g. $(1 + \lambda^2)y + 2\lambda x - 1 = 0$.

A 29 • ↓ **A 32**

Let $f(x)$ be a 2-periodic extension of $1 - |x|$, $x \in [-1, 1]$. Let C_n be the graph of $f_n(x) = 2^{-n} f(2^n x)$, $x \in [-1, 1]$. Show that the length of C_n is $2\sqrt{2}$ for all integers n.
Prove that the maximal distance between C_n and points $x \in [-1, 1]$ tends to zero, i.e. a curve of length $2\sqrt{2}$ cannot be reasonably distinguished from a curve of length 2. What is wrong here?

A 30 •• ↓ **A 31** ↓ **A 33** ↓ **A 34**

Consider an increasing smooth function $s = g(t)$, mapping the interval $(0, 1)$ onto $(0, a)$. Denote by g_{-1} its inverse function. Two smooth arcs with parametric equations $x = x(t)$, $y = y(t)$, $t \in (0, 1)$ and with equations $x = x(g_{-1}(s))$, $y = y(g_{-1}(s))$, $s \in (0, a)$ describe the same set of points. Find a function g, the value of which at the point t gives the length of the arc between the points $(x(0), y(0))$ and $(x(t), y(t))$. The foregoing parametrization is called parametrization by arc length.

8

A 31 • [M] ↑ **A 30**

Find the parametrization by arc length of the curves given by $y = \cosh x$ starting at $x = 0$ and $y = \log(\cos x)$ starting at $x = \pi/2$.

A 32 • ↑ **A 29**

Suggest and analyze various ways how to measure the length of a river.

Hint: Can a newly built dam or a heavy rainfall make the river longer?

A 33 •• ↑ **A 30** ↓**A 34**

On a parabola, two points (x_1, y_1) and (x_2, y_2) are given. Find a point (x_0, y_0) which divides the arc between (x_1, y_1) and (x_2, y_2) into two parts of equal length.

A 34 •• ↑ **A 30** ↑ **A 31** ↑**A 33**

Can you describe a method of finding n points equally spaced on a simple arc? Can you give a reasonable and simple approximation of these points?

A 35 • ↑ **A 24**

A closed simple arc defines a domain. Express the area of this domain by a curvilinear integral along the arc.

Hint: Specialize Green's formula.

A 36 ••• ↓ **PA 36** ↑ **A 35**

Given the length d of a simple closed planar arc A, denote the enclosed area $P = P(A)$. Prove that P attains its maximum for A being a circle of length d.

A 37 •• ↓ **A 38**

A motionless vessel at the point A is pulled on a rope of constant length along a straight line (which does not contain the point A) at a constant speed. Find the curve on which the pulled vessel will move.

Hint: Let the straight line be the positive part of the x-axis and $A = (0, a)$, $a > 0$. The rope of constant length forms a tangent to the curve we are looking for. Assume the curve to be the graph of a function $y = y(x)$ and find the differential equation which has to be satisfied by y.

A 38 ••• ↓ **PA 38** ↑ **A 37**

When the front wheel of a bike moves along a path $\zeta = \zeta(t)$, $\eta = \eta(t)$, $0 \le t \le T$, the rear wheel follows, in general, another path. They are identical iff the front wheel moves along a straight line.

1. Determine the path of the rear wheel in other cases or find the path of the front wheel when the other one is given.
2. Given the two paths above, can it be decided in which direction the bike moved?

A 39 •• ↑ **A 38**

Find the path of the rear wheel of a bike, if the front wheel follows the path $\zeta = 2.5 \cos t$, $\eta = 2.5 \sin t$, $t \in [0, \pi]$ and the rear wheel starts at the position (x, y) with $x = 1.8$, $y < 0$ and at the distance 1 from the initial position of the front wheel.

A 40 ●● ↓ PA 40↓ A 41 ↓ A 42

A target A is moving along a known path P at a constant speed a. From the point B an object is moving and directed at any moment towards the target with the constant velocity v. Determine the curve on which the object will move, assuming that the path P is a straight line.

Hint: Let the straight line be the x-axis. Assume the curve to be the graph of a function $y = y(x)$ and find the differential equation which has to be satisfied by y.

The curve in A 37 is called a tractrix. In A 40 – A 42 the resulting curves are called pursuit curves.

A 41 ●●● ↓ PA 41 ↑ A 40 ↓ A 42

A target A is moving along a known path P, $A = (\xi(t), \eta(t))$, t means time. From the point B an object starts moving and it is directed at any moment towards the target. Determine the curve on which the object will move.

A 42 ●● ↑ A 41

Let the target A be moving as $A = (t, -t^2)$, $t \geq -1$. Determine the pursuit curve of the object starting from the origin and with velocity equal to the $5\sqrt{1+4t^2}$ multiple of the distance to A at time t.

Supplementary Material

Definitions

An arc is a set $A \subset \mathbf{R}^n$ such that there is a continuous mapping $\varphi : [a, b] \to A$ onto A such that

(i) φ is piecewise C^1, i.e. it has a piecewise continuous derivative;
(ii) $\varphi(s) \neq \varphi(t)$ for all $s, t \in [a, b]$ $s \neq t$ with the only possible exception for $(s, t) = (a, b)$;
(iii) the derivative φ' does not vanish at any $t \in [a, b]$.

The points $\varphi(a)$, $\varphi(b)$ are called the endpoints of the arc.
A curve is commonly considered to be a finite succession of arcs such that for any two successive arcs the endpoint of the predecessor coincides with the initial point of the successor. A curve is called simple if it has no self-crossing.
An arc is closed if its endpoints coincide.

Let $\varphi = (\varphi_1, \varphi_2)$ be an arc in $\mathbf{R}^2$, $t \in [a, b]$. The vector φ' (if it exists) is called its tangent vector. For any $t_0 \in [a, b]$ the set of points

$$x(t) = (t - t_0)\varphi_1'(t_0) + \varphi_1(t_0), \qquad y(t) = (t - t_0)\varphi_2'(t_0) + \varphi_2(t_0),$$

for $t \in \mathbf{R}$ is called its tangent at t_0.
For $t \geq t_0$ (or $t \leq t_0$) this set is called its half-tangent.

The set of points (x, y) satisfying the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{23}x + 2a_{13}y + a_{33} = 0$$

is called a conic. If written in the form $\{x, y, 1\}A\{x, y, 1\}^T = 0$, the 3×3 symmetric matrix A is called the matrix of the conic. If $\det A = 0$ the conic is called singular, otherwise it is called regular.
A regular conic with $\Delta = a_{11}a_{22} - a_{12}^2$ is called a parabola, ellipse, hyperbola, if $\Delta = 0$, $\Delta > 0$, $\Delta < 0$, respectively.

If C is an arc in $\mathbf{R}^n$ with $\varphi = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ then the integral

$$\int_a^b \sqrt{\dot{\varphi_1}^2 + \dot{\varphi_2}^2 + \cdots + \dot{\varphi_n}^2}dt$$

is equal to the length of the arc C.

Plans of Solution

PA 19

1. The motion is described in spherical coordinates

$$x = r\sin\theta(t)\cos\varphi(t), \qquad y = r\sin\theta(t)\sin\varphi(t), \qquad z = r\cos\theta(t)$$

with $\theta(t), \varphi(t)$ unknown functions of time.
The velocity condition is then

$$|v|^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta).$$

2. The tangent vector to the constant latitude ($\theta(t) = \text{const}$) is

$$w = (-r\sin\theta(t)\sin\varphi(t)\dot{\varphi}(t),\; r\sin\theta(t)\cos\varphi(t)\dot{\varphi}(t), 0).$$

Hence

$$\cos\alpha = \frac{w \cdot v}{|w||v|} = \frac{r}{|v|}\sin\theta(t)\dot{\varphi}(t).$$

3. To obtain the description of the trajectory combine the two equations above to get $\dot{\phi}^2\sin^2\theta\tan^2\alpha = \dot{\theta}^2$, i.e.

$$\frac{d\theta}{d\varphi} = \pm\sin\theta\tan\alpha.$$

4. The solution of the last equation is $\theta(\varphi) = 2\arctan(e^{\pm\varphi\tan\alpha})$, which gives two curves

$$x = r\sin\theta(\varphi)\cos\varphi, \qquad y = r\sin\theta(\varphi)\sin\varphi, \qquad z = r\cos\theta(\varphi).$$

5. To get the position of the airplane in time t, eliminate $\dot{\varphi}$ from equations in (1) and (2) to get $\theta(t) = \pm t\frac{|v|}{r}\sin\alpha + \theta_0$, where θ_0 is the position in $t = 0$. Then equation in (2) gives

$$\varphi(t) = \pm\cot\alpha\log\tan\frac{\pm t|v|\sin\alpha + \theta_0}{2}$$

for $\alpha \neq 0$,

$$\varphi(t) = \frac{t|v|}{r\sin\theta_0} \quad \text{for } \alpha = 0.$$

6. The time needed to reach the North (South) pole starting at the latitude θ_0 is $t = \frac{\theta_0}{|v|\sin\alpha}$ $(t = \frac{\pi - \theta_0}{|v|\sin\alpha})$.
7. The problem can alternatively be solved applying the second result of M 20 since the Mercator projection of the path is a straight line.

PA 28

1. We may assume that the curve is parametrized in such a way that $(x(\lambda), y(\lambda))$ is the tangent point lying on the line $y = k(\lambda)x + q(\lambda)$. Hence, $y(\lambda) = k(\lambda)x(\lambda) + q(\lambda)$.
2. Notice that $\frac{y'(\lambda)}{x'(\lambda)} = k(\lambda)$. (Derivatives with respect to λ.)
3. Differentiate the equation in step 1 and substitute for $y'(\lambda)$ from step 2.
4. Obtain two equations

$$\begin{aligned} y(\lambda) &= k(\lambda)x(\lambda) + q(\lambda), \\ 0 &= k'(\lambda)x(\lambda) + q'(\lambda) \end{aligned}$$

for the unknown $x(\lambda)$, $y(\lambda)$. You may also eliminate λ from the two equations so as to obtain the relation between x and y.

PA 36

1. Assume that the arc is given by a complex-valued function $z = z(t)$. Without loss of generality we may assume that $|z'(t)| = 1$, i.e. that the arc is parametrized by its length. Hence $z(0) = z(d)$.
2. Green's formula yields for the area $A = \frac{1}{2}\operatorname{Im}\int_0^d z'(t)\bar{z}(t)dt$. For two complex functions $u, v : [0, d] \to \mathbf{C}$ Parseval's identity reads

$$\int_0^d u(t)\bar{v}(t)dt = d\sum_{k=-\infty}^{\infty} U(k)\overline{V}(k),$$

where $U(k)$, $V(k)$ is the k-th Fourier coefficient of u, v, respectively. Therefore

$$A = \pi\sum_{k=-\infty}^{\infty} k|Z(k)|^2,$$

where $Z(k)$ denotes the k-th Fourier coefficient of $z(t)$.
3. Similarly, $d = \int_0^d |z'(t)|^2 dt = \frac{4\pi^2}{d}\sum_{k=-\infty}^{\infty} k^2|Z(k)|^2$. Hence,

$$\frac{d^2}{4\pi} - A = \pi\frac{4\pi^2}{d}\sum_{k=-\infty}^{\infty}(k^2 - k)|Z(k)|^2.$$

4. Since the right-hand side of the last equation is nonnegative, the maximum area is $\frac{d^2}{4\pi}$ and it is achieved only if $k^2 - k = 0$, i.e. $k = 0, 1$. But this is possible iff $z(t) = Z(0) + Z(1)\exp(2\pi\mathbf{i}t/d)$, which represents a circle.

8

PA 38

1. (a) Assume that the distance between the rear and front wheels equals 1. Denoting by $(x(t), y(t))$ the parametrization of the unknown path ρ of the rear wheel we have

$$(x-\zeta)^2 + (y-\eta)^2 = 1, \quad \text{for all } 0 \le t \le T.$$

   (b) Since the tangent to ρ at any point is directed towards the actual position on the path of the front wheel, and the velocities on both curves should be equal, we obtain

$$(\dot{x}, \dot{y}) = \frac{(\zeta - x, \eta - y)}{\|(\zeta - x, \eta - y)\|} \|(\dot{\zeta}, \dot{\eta})\| \quad \text{for all } 0 \le t \le T.$$

   Here, the denominator equals 1.

   (c) This is a system of two differential equations. The unknown (ζ, η) given the initial conditions $(x(0), y(0))$ should again satisfy the condition $(x-\zeta)^2 + (y-\eta)^2 = 1$ for $t = 0$.

   (d) Similarly, for (x, y) given, the same condition must hold true for $(\zeta(0), \eta(0))$.

2. Any tangent to the path ρ crosses the other path in two points. Find the one which is at a given distance from the tangent point and decide on the direction of the movement.

PA 40

1. The position of the moving target is $X = x_0 + at$, $Y = 0$. Denote the coordinates of the object by x, y and assume that the object is moving at a constant speed v. We obtain $\dot{x}^2 + \dot{y}^2 = v$ where the dots denote differentiation with respect to t.
2. At a position (x, y) the tangent to the curve should point towards the point $x_0 + at$, hence $\frac{dy}{dx} := y' = \frac{-y}{X-x}$ or
$$x_0 - x + at = -\frac{y}{y'}. \tag{*}$$
3. We want to exclude the variable t. From the equation which determines the speed v of the object we obtain $\frac{dt}{dx} = \frac{1}{v}\sqrt{1+y'^2}$ and taking the derivative with respect to x (indicated by primes) of the equation in (*) we obtain $\frac{dt}{dx} = \frac{yy''}{ay'^2}$. Comparing the two last results we conclude: The pursuit curve satisfies (in this setting) the differential equation
$$y'' = \frac{a}{v}\frac{y'^2}{y}\sqrt{1+y'^2}.$$
4. Putting $y' = p(y)$ we obtain a differential equation for the unknown function $p(y)$, where variables can be separated. Adjusting the initial conditions, i.e. taking into account that at the initial position (x_0, y_0) of the object $1/p = 0$, we obtain a differential equation for $y = y(x)$ as follows
$$-\frac{dx}{dy} = \frac{1}{2}\left(\left(\frac{y}{y_0}\right)^{a/v} - \left(\frac{y}{y_0}\right)^{-a/v}\right)$$
which can easily be solved.
5. If $a = v$ we obtain
$$x = \frac{y_0}{4}\left(\left(\frac{y}{y_0}\right)^2 - 1 - 2\log\left(\frac{y}{y_0}\right)\right).$$
In this case the object can never reach the target.
6. If $a < v$ then some calculation yields that the object reaches the target at time
$$T = \frac{y_0 v}{v^2 - a^2}.$$

8

PA 41
With P defined by $\xi = \xi(t)$, $\eta = \eta(t)$ we obtain for the unknown path $(x(t), y(t))$

$$\dot{x}(t) = \nu(t)(\xi(t) - x(t)),$$
$$\dot{y}(t) = \nu(t)(\eta(t) - y(t))$$

where $\nu(t)$ is an arbitrary nonnegative function. The velocity of the object is proportional to the distance of the target and object and $\nu(t)$ is the proportionality factor. This proportionality factor could be e.g. a λ multiple of the velocity of the target. The actual value of the velocity of the object is

$$\nu(t)\sqrt{(\xi(t) - x(t))^2 + (\eta(t) - y(t))^2}$$

and it can be subject to additional conditions.

Center of Mass and Moments 9

The dynamics of an isolated system of N free particles, each with mass m_i, $i = 1, 2, \dots, N$, is completely determined by the collection of their moments $p_i = m_i v_i$, where v_i are vectors of velocities of the corresponding particle. The total momentum of the system $P = \sum_{i=1}^{N} m_i v_i$ can be represented as the momentum of a single point with mass $M = \sum_{i=1}^{N} m_i$ moving with the velocity V such that $P = MV$. This single point is called the center of mass of the system. It is also called the center of gravity or the centroid of the system.

The center of mass is a mathematical concept, its physical meaning shows one of the most important features of mathematical models. When mathematical models are defined in a certain coordinate systems then the physically interpretable results must be independent of this coordinate system. Such independence should be proved, otherwise the mathematical model cannot be applied to the physical (or other) phenomenon.

Other physical problems (e.g. rotation of a system of points) lead to generalizations. The sum $\sum q_i d_i^k$, where q_i are weights, d_i are distances, usually from a given fixed point or line, is called the k-th moment of the system. Applications of this concept in probability theory and other parts of mathematics were a basis for a theory of moments which found its use also in system theory, approximation theory, theory of integral equations etc.

> This chapter illustrates the use of some mathematical methods in physics. It also shows how solution of applied problems enforces and inspires further development of mathematical concepts.
> The solution of problems marked as [M] and many others can be made easier when using software packages comparable to Maple® or S. Wolfram's MATHEMATICA®, although only very few of them can be solved directly. The user is strongly advised to learn the syntax and semantics of some of the com-

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mands. In this chapter the following commands and packages (and related ones) may help:

In MATHEMATICA®:
Packages named `Statistics`... ``,
Commands `Min, Plus, LegendreP, ContourPlot`

In Maple®:

`Statistics, OrthogonalSeries, plots` packages and
Commands `min, LegendreP`

In most of the examples numerical experiments may give a starting point for reasoning or verify initial conjectures. Explanation of mathematical terms and concepts can be found on the Internet, e.g. at www.mathworld.wolfram.com or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://eom.springer.de.

Suggestions

- Easy starting points are examples numbered C 01 to C 06 and C 10, C 14, C 21.
- Examples C 02, C 15, C 31, C 32 show how important it is to choose a suitable coordinate system.
- Outlooks to some special parts of mathematics are shown in C 22, C 23, C 34, C 43.
- Examples C 24 to C 27 are closely connected to probability theory.
- All readers are strongly encouraged to modify, generalize or simplify the formulated problems, to find alternative formulations and formulate, solve their own examples and compare the context of these problems with the given ones.

Problems

C 01 • ↓ **C 02** ↓ **C 03** ↓ **C 14** ↓ **C 16**

Let $A_1, A_2, \dots, A_n$ be points in $\mathbf{R}^N$ with corresponding masses q_i. Find the coordinates of their center of mass in the Cartesian coordinate system.

Hint: In $\mathbf{R}^2$ the position can also be given by a complex number. In that case a complex number z can be represented as $z = r(\cos\alpha + \mathbf{i}\sin\alpha)$, where $\mathbf{i}^2 = -1$.

C 02 • ↑ C 01 ↓ C 31 ↓ C 32

Show that the center of mass is invariant under the change of the inertial system of coordinates, i.e. with the new coordinates x' given by $x' = Ax$ where A is an $n \times n$ regular matrix and $x = (x_1, x_2, \dots, x_n)$, $x' = (x'_1, x'_2, \dots, x'_n)$ or by $x' = x + b$, $b = (b_1, b_2, \dots, b_n)$.

C 03 • ↑ C 01 ↓ C 14

(a) Let T be the center of mass of a system of n points of equal mass q. When a point A_{n+1} of mass q is added to this system, the center of mass of the $n+1$ points will move to another point T^+.

Find the vector $T^+ - T$ by which the original center of mass has moved.

(b) Find $T^+ - T$ also for systems of n points with unequal masses q_i, $i = 1, 2, \dots, n+1$.

C 04 • ↑ C 01 ↑ C 02

If a set of points of equal mass in $\mathbf{R}^2$ is symmetric with respect to a straight line l, then this line contains their center of mass. Can you prove this statement? How would you formulate and prove this statement for points in $\mathbf{R}^3$?

C 05 • ↓ C 34

Suppose there are two finite sets A, B of points in $\mathbf{R}^N$ with overall masses equal to q_A and q_B, respectively. Let r_A denote the center of mass of A and similarly r_B for B. Find the center of mass of the set $A \cup B$. Generalize your conclusion to a union of a finite number of finite sets. Reconsider your conclusion when the sets A, B are not finite sets.

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C 06 • ↑ C 01

Find all polynomials of degree n such that the center of mass of their roots (each root represents a point of unit mass) is at the origin of the complex plane. Find a transformation of the variable yielding a polynomial with the center of mass of its roots at the origin.

The transformation asked for in C 06 is the first step in deriving formulae expressing the roots of quadratic, cubic (Cardano's formula) and quartic (Euler's formula) polynomials in terms of the coefficients.

C 10 • ↓ C 11 ↓ C 12 ↓ C 13 ↓ C 18

Place two points of equal mass on a circle so that their center of mass is at a prescribed fixed point inside of the circle. Is the solution unique?

C 11 •• ↑ C 10 ↓ C 16

Place three points of equal mass on a circle so that their center of mass is at a prescribed fixed point inside of the circle. Is the solution unique?

C 12 ••• ↓ PC 12 ↑ C 03 ↑ C 10 ↑ C 11 ↓ C 18

(a) Prove that n points of equal mass can be placed on a circle so that their center of mass is at a prescribed fixed point inside of the circle.
(b) For a prescribed point find an algorithm placing the n points on the circle.
(c) Apply the algorithm to the unit circle and the prescribed point $(\rho, 0), 0 \leq \rho < 1$, with $n = 6$ and $n = 7$ points, respectively.

C 13 • [M] ↑ C 10

If possible, place two points of mass p, q respectively, on a circle so that their center of mass is at a prescribed fixed point inside the circle. Find conditions under which this problem has no solution.

Hint: If $p = q$ then the solution is simple.
If $p \neq q$ note that the given point must lie at a uniquely determined place on the chord of the circle joining these two points.

C 14 • ↑ C 01 ↑ C 03 ↑ C 05

A slightly nonhomogeneous circle of mass q and radius r has its center of mass displaced from the center of the circle to the distance of $r/20$. We want to place a point of mass m on the circle so that the center of mass of the resulting circle would be at its center. Find the mass m and the point where it is to be placed.

C 15 • • • [M] ↑ C 01

A set of n points with masses $q_i = q + \epsilon_i$, $i = 1, 2, \ldots, n$ have to be placed on a circle at fixed equidistant points. All the ϵ_i are small compared with q. Find their order so that the center of mass of the set is as close to the center of the circle as possible. Can you estimate the complexity of your solution?

Hint: You can solve actual moderate-size examples examining all the permutations. In MATHEMATICA® e.g. in the following way:

```
mass = 1.004, 1.001, 1, 0.997, 0.995
places = Table[Exp[2 Pi k I/5], k, 0, 4]
Min[Abs[Table[Apply[Plus,
N[mass*Permutations[places][[k]]] ], k, 1, 120]]] //
Timing
```

This shows the best possible result. It still remains to determine which of the permutations is the best one. To this end a modification of the command Min could be used. Try to design such a procedure.

C 16 • ↑ **C 10**

Let there be given a simple closed planar curve such that the origin lies in its interior. Prove that two points of equal mass can always be placed on the given curve so that their center of mass will be the origin.

Hint: In the proof you might find it helpful to use a theorem on functions which are continuous on a closed interval (see the Theorems below). The function to be considered here is the ratio between the two parts of a chord passing through the origin.

C 17 •• ↑ **C 02** ↑ **C 10** ↑ **C 12** ↑ **C 16**

Let there be a simple closed planar curve such that the origin lies in its interior. Prove that n points of equal mass can always be placed on the curve so that the center of mass of these n points lies at the origin.

Hint: Try to perform a proof by induction on n.

C 18 • ↑ **C 12**

Place n points of equal mass on a given ellipse so that their center of mass lies in a given interior point of the ellipse.
Solve for $n = 6$, ellipse $x^2 + 3y^2 = 3$ and the point $(1/12, 1/12)$.

Hint: To use the method of C 12 you may find useful the following concept of conjugate diameters of an ellipse.
Any chord passing the center of the ellipse is called its diameter. Any diameter C of the ellipse specifies a family of parallel chords. The midpoints of all these chords define another diameter, which is called the diameter conjugate to C.

C 21 **•• [M]** ↓ **C 22**

Denote the moments of order $k = 0, 1, 2, \ldots$ of a polynomial of degree n on the interval $(0, 1)$ by $m_k^{(n)}$.
Find conditions for n, k such that the set of moments uniquely determines the polynomial. Find a method for recovering the polynomial from its given moments.

C 22 **•• [M]** ↑ **C 21** ↓ **C 23**

Consider a function $f : (a, b) \to \mathbf{R}$ and its first n moments on this interval. Find a polynomial P of degree $n - 1$, which has equal moments with respect to (a, b). Can this polynomial be used as an approximation of the function f?
Try e.g. $f(x) = \sin \pi x$, $-1 \le x \le 1$, $n = 6$.

Hint: You will have to find the inverse of the Hilbert matrix, which is known to be ill-conditioned. (The matrix $[a_{ik}]$ of order n with elements $a_{ik} = \frac{1}{i+k}$, $i, k > 0$ is called the Hilbert matrix.)

C 23 **• [M]** **X 07** ↑ **C 22**

Suppose the function f is given on the interval $[-1, 1]$. Find the n-th partial sum of its orthogonal series of Legendre polynomials and compare the result with those of C 22.

Hint: The Legendre polynomials of degree n can be found by the command `LegendreP[n,x]` in MATHEMATICA® or `with (orthopoly);P(n,x)` in Maple®.

C 24 • [M] ↓C 26 ↓ C 27

Compare the behavior of the function

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{s^2 - q^2}} \exp \frac{-(x-a)^2}{2(s^2 - q^2)}$$

for various values of $q, s, q \geq 0$.
Find its first moment and the second central moment. Compare the characteristic functions of f with two different values of q.

Hint: Find the ration of the two characteristic functions.

C 25 • ↑ C 23

Express the k-th central moment of a function on an interval I in terms of moments of order $j \leq k$.
For $k = 2, 3, 4, 5$ find formulae expressing the k-th moment in terms of central moments of corresponding order.

C 26 •• [M] ↑ C 25

(a) Give examples of unimodal functions. Find some of the probability density functions listed in the standard packages as Statistics ‘Continuous Distributions’ which are unimodal. Find the moments of these functions.
(b) Give an interpretation of the inequalities $m_2(f) < m_2(f)$ if f, g are unimodal functions such that their moments on the real axis satisfy the conditions $m_0 = 1$, $m_1 = 0$.
(c) Find an interpretation of the third (and fourth) central moment.

C 27 •• [M] ↑ C 24 ↑ C 25 ↑ C 26

As a result of measurements the following data have been obtained

```
{{-1.5, 0.02999}, {-1.3, 0.02516}, {-1.1, 0.00755},
  {-0.9, 0.02357}, {-0.7, 0.04319}, {-0.5, 0.03689},
  {-0.3, 0.05383}, {-0.1, 0.06105}, {0.1, 0.07341},
  {0.3, 0.13566}, {0.5, 0.14249}, {0.7, 0.19496},
  {0.9, 0.25378}, {1.1, 0.29706}, {1.3, 0.32772},
  {1.5, 0.40111}, {1.7, 0.40535}, {1.9, 0.4348},
  {2.1, 0.39781}, {2.3, 0.40223}, {2.5, 0.40052},
  {2.7, 0.35724}, {2.9, 0.30408}, {3.1, 0.25794},
  {3.3, 0.19071}, {3.5, 0.15106}, {3.7, 0.12776},
  {3.9, 0.08809}, {4.1, 0.08055}, {4.3, 0.03051},
  {4.5, 0.02574}, {4.7, 0.01583}, {4.9, 0.03472},
  {5.1, 0.01384}, {5.3, 0.04697}, {5.5, 0.03269}}.
```

A plot suggests that these data may originate from a normally distributed random variable. Find the density function of this random variable.

Hint: Find the first moment and the second central moment of the data. Compare the results of measurements with the normal distribution possessing the same moments, remembering that a normal distribution is uniquely determined by these moments.

C 31 •• ↑ C 02

A homogeneous arc of a unit circle with its central angle α and mass $q\alpha$ is freely suspended at one of its ends. Find the coordinates of its other end.
Solve this problem for $q = 1$, $\alpha = \pi/3$ and $\alpha = \pi/2$.

Hint: Find the center of mass of the arc and then perform a transformation of the coordinate system so that one end of the arc is at the origin and the center of mass is at the negative part of the y-axis.
The calculations are easier to perform in the complex plane.

9

C 32 ●● ↑ **C 02**

A glass has the shape of a cylinder with diameter r and height h and is full of water. If the glass is tilted by an external force to an angle α some of the water spills. If the glass is than left free, it either returns to its original position or it falls down. Determine the critical value of the angle α, which separates the two cases.

Hint: The critical value of α is when the projection of the center of mass of the tilted glass onto the horizontal plane coincides with the point where the tilted glass touches the plane. It is convenient to choose the origin of the coordinate system at this point.

C 33 ●● ↓ **PC 33**

Prove Gulden's rule formulated as follows:
The area P of a surface formed by rotating a planar curve around an axis which has no common points with the curve is equal to the product Sd, where S is the length of the curve and d is the length of the circle circumscribed by the center of mass of the curve.

C 34 ● ↑ **C 05**

Let the mass of an ideal rod of length l have density $\rho(x) > 0$, $0 \le x \le 1$. Moreover, let there be masses q_i attached to the rod at points c_i, $i = 1, 2, \dots, n$. Find the center of mass of the rod.

Remark: Put $f(x)$ equal to the mass of the part of the rod lying in the interval $[0, x]$. Since f is non-decreasing there is an $F(x)$ such that $F'(x) = f(x)$ except for the values $x = c_i$, $i = 1, 2, \dots, n$.

The mass of the rod can be expressed as $m = \int_0^l \rho(x) dF(x)$.

Considerations similar to those of C 34 and others in probability theory led to the concept of the Stieltjes integral. The Stieltjes integral plays an important role in probability theory, measure theory, theory of moments and other parts of mathematics.

C 41 • ↑ C 01

Let A_i, $i = 1, 2, \dots, n$ be points in $\mathbf{R}^3$. Find the point B which minimizes the sum of squares of distances from A_i to B.

C 42 •• **[M]** **V 09** ↑ C 01 ↑ C 41

For the points $(0, 0)$, $(0, 1)$, $(1, 0)$ find the point B which minimizes the sum of distances between B and the given three points. Compare the result with C 41.

Hint: The sum of distances is a continuous function on the triangle defined by the three points. It attains a minimum. Due to symmetry, the minimum must be attained at a point with equal x and y coordinates. It may help to plot the graph of the function f to be minimized.

C 43 • ↑ C 41

Let V be a solid of homogeneous density $\rho = 1$. Prove that the function

$$g(\xi, \eta, \zeta) = \iiint_V [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]\, dV(x, y, z)$$

attains its minimum at the center of mass of V.

Hint: Verify whether the integral can be differentiated with respect to parameters ξ, η, ζ.

C 44 •• ↑ C 41 ↑ C 43

Is the center of mass of a homogeneous arc $x = x(t)$, $y = y(t)$, $z = z(t)$, $0 \le t \le 1$ of unit density at the point where the function

$$h(\xi, \eta, \zeta) = \int_0^1 [(x(t)-\xi)^2 + (y(t)-\eta)^2 + (z(t)-\zeta)^2] dt$$

attains its minimum?

If your answer is ‘yes’ then find a proof, if your answer is ‘no’ then give the correct result, i.e. express the center of mass of an arc as the minimum of a certain integral depending on ξ, η, ζ as its parameters.

9

Supplementary Material

Definitions

The product qd, where q is the mass of a point $A \in \mathbf{R}^n$, and d is the distance of A to a fixed point (straight line, plane), is called the static moment of A with respect to that point (straight line, plane). The static moment of a set of points is defined as the sum of static moments of all points of the set.
The product qd^2 is (in a similar manner) called the moment of inertia.

In a set A of points of overall mass Q the center of mass of A is defined as the point of mass Q whose static moment is equal to the static moment of A.

Let $A \in \mathbf{R}^n$ be a body with mass density ρ. Then the coordinates x_i, $i = 1, 2 \ldots, n$ of its center of mass satisfy the following equation

$$x_i \int_A \rho dV = \int_A x\rho dV, \quad i = 1, 2 \ldots, n.$$

For a function f the values

$$m_k(f) = \int_a^b x^k f(x)dx, \quad k = 1, 2 \ldots$$

are called moments of order k of the function f on the interval (a, b) provided that the integral exists.

The integrals

$$\mu_k(f) = \int_a^b (x - m_1(f))^k f(x)dx, \quad k = 1, 2, \ldots$$

are called central moments of f with respect to (a, b).

Let ρ be a positive function on $[a, b]$ and let φ_n, $n = 1, 2, \ldots$ be a sequence of real functions on $[a, b]$ such that

$$\int_a^b \varphi_n(x)\varphi_m(x)\rho(x) = 0 \quad \text{for } n \neq m.$$

Then $\{\varphi\}$ is said to be an orthogonal system of functions on $[a, b]$ with weight ρ.
For a function f on $[a, b]$ the series $\sum_{n=1}^{\infty} c_n\varphi_n(x)$ with

$$c_n = \frac{1}{\int_a^b \varphi_n^2(x)dx} \int_a^b f(x)\varphi_n(x)dx$$

is called the orthogonal series of f (relative to φ_n).

A positive function $f : \mathbf{R} \to \mathbf{R}$ is called unimodal if it has exactly one strict maximum and if it is strictly monotonic at all other points.

If f is a probability density function then the function

$$\Phi(t) = \int_{-\infty}^{+\infty} f(x) \exp(\mathbf{i}\, tx) dx$$

with $\mathbf{i}^2 = -1$ is called its characteristic function.

Polynomials defined by the formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

are called Legendre polynomials.

Theorems

A

Let f be a continuous real function on a closed interval I, $a < b$, and $a, b \in I$.

- If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there is a point $x \in [a, b]$ such that $f(x) = c$.
- There are two points $u, v \in [a, b]$ such that $f(x) \leq f(u)$ and $f(x) \geq f(v)$ for all values $x \in [a, b]$, i.e. such function attains both its maximum and minimum value.

B

The basic linear distance-preserving transformations of the planar Cartesian coordinate systems are rotation and translation. Rotation by the angle t is given by

$$\xi = x \cos t + y \sin t, \quad \eta = -x \sin t + y \cos t,$$

and translation to a new origin (m, n) is given by

$$\xi = x - m, \quad \eta = y - n,$$

where (ξ, η) are the new coordinates.

C

Legendre polynomials form an orthogonal system on $[-1, 1]$ with weight $\rho(x) = 1$, i.e. there is

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{2}{2n+1} & \text{if } n = m. \end{cases}$$

9

Plans of Solution

PC 12
(a) Such configuration exists. This can be proved by induction. For $n = 2$ see C 10. Assume that the configuration of n points with given center of mass exists. Having $n+1$ points, use the results of C 02 to pre-calculate the position of the center of mass of n points in such a position that with adding the last $(n+1)$-st point we end up in the prescribed place.
(b) In the complex plane with the circle $|z| = 1$ and the given center of mass at $z = \rho$, $0 < \rho < 1$ any two complex conjugate points $z_0, \bar{z_0}$ have their center of mass on the real axis. Therefore adding such a symmetric pair of points, the center of mass c an be moved. Start with two or three points for n even and odd, respectively.

PC 33

1. Choose the coordinate system so that the x-axis is the axis of rotation and the equation of the curve reads $y = f(x)$. Then f is a nonnegative function, defined for $a \le x \le b$.
2. Express the area P by
$$2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx.$$
3. Find the y coordinate of the center of mass of the curve.
4. Compare the last two results.

Further References

Approximations of the type of Example C 23 have been widely developed in the theory of orthogonal polynomials. There is a vast selection of books, one of the classical and best known is

Szegö G., Orthogonal Polynomials, Am. Math. Soc., New York, 1939.

The functions considered in Example C 24 with $q > 0$ are called the Gauss transform of the normal density function. The use of this Gauss transform is motivated by properties of Fourier integrals of the normal probability density function (i.e. its characteristic function) and its Gauss transform.

Far reaching development of C 21 and C 22 is given e.g. in

Akhiezer N.I., The Classical Moment Problem and Related Questions in Analysis, Hafner, 1965.

Miscellaneous 10

In this chapter we give some examples from various fields of applications in which ad hoc methods are applied. They serve merely as illustrations showing that solutions of some applied problems may demand rather unexpected ways of mathematical reasoning.

Problems

X 01 •

A real polynomial P assumes only nonnegative values if and only if there exist two real polynomials M, N such that $P = M^2 + N^2$.

Hint: Notice that the degree of P is even and use induction on the degree with the use of the identity

$$(p_1^2 + q_1^2)(p_2^2 + q_2^2) = (-p_2 q_1 + p_1 q_2)^2 + (p_1 p_2 + q_1 q_2)^2.$$

X 02 • **M 11**

Find $y(x)$ such that $x^2 + 2xy(x) + y(\frac{1}{x}) = 0$ for all $x \neq 0$.

Hint: Replace x by $1/x$.

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X 03 • **P 26 P 27 M 11**

Find a solution y for the equation

$$2xy(x^2) + y\left(\frac{x^2+1}{x^2-1}\right) + 1 = 0$$

with $x > 1$.

Hint: Change x to $\sqrt{\frac{x^2+1}{x^2-1}}$.

The above approach to some functional equations can be slightly generalized as follows: Let ψ be a 'self-inverse' function on the interval I, i.e. $\psi(\psi(x)) = x$ and let ϕ have an inverse on the interval I. Then the functional equation for the unknown function $y(x)$ in the form $F(x, y(\phi(x)), y(\psi(\phi(x))) = 0$ can be rewritten in two ways putting $u = \psi(\phi(x))$

$$F(\phi^{-1}(\psi(u)), y(\psi(u)), y(u)) = 0,$$
$$F(\phi^{-1}(\psi(u)), y(u), y(\psi(u))) = 0,$$

where ψ^{-1} denotes the inverse of ψ. Excluding $y(\psi(u))$ from these two equations, we can obtain a solution y. In X 03 we had $I = (1, \infty)$, $\psi(x) = \frac{x+1}{x-1}$, $\phi(x) = x^2$.

X 04 • **P 26 P 27 M 11**

Let $I = [0, 1]$. Find, if it exists, a continuous solution $y : I \to \mathbf{R}$ of the following equations:

1. $y(x) = y(\sqrt{1-x^2})$
2. $\sin(y(\sqrt{1-x^2})) = 2\cos y(x)$
3. $\sinh(y(\sqrt{1-x^2})) = 2\cosh y(x)$

X 05 •

A function f is defined as follows

$$f(x) = \sum_{k=1}^{n} \frac{A_k}{x - a_k}$$

with $A_k > 0$, $0 < a_1 < a_2 < \cdots < a_n$. Show that the function $f(x)$ has $n - 1$ real zeros $b_1, b_2, \ldots, b_{n-1}$ and that these zeros interlace the zeros of the denominator, i.e. $a_1 < b_1 < a_2 < b_2 < \cdots < b_{n-1} < a_n$.

Hint: Express the numerator of f in terms of a_k.

X 06 • ↑ **X 05**

A function f is defined as follows

$$f(x) = \sum_{k=1}^{n} \frac{A_k x}{x^2 + a_k^2}$$

with $A_k > 0$, $0 < a_1 < a_2 < \cdots < a_n$. Show that the function $f(x)$ has $n - 1$ pure imaginary zeros $b_1, b_2, \ldots, b_{n-1}$ and that $a_1^2 < -b_1^2 < \cdots < -b_{n-1}^2 < a_n^2$.

X 07 •• **C 23** ↑ **X 05**

Consider a set of polynomials P_n defined by the recurrence

$$(n+1)P_{n+1}(x) = (2n+1)x P_n(x) - n P_{n-1}(x),$$

$$P_0(x) = 1, \qquad P_1(x) = x.$$

Assuming that $P_n(a) = P_n(b) = 0$ for some real values a, b prove that there exists a real value c, $a < c < b$, such that $P_{n+1}(c) = 0$, i.e. between any two consecutive real zeros of P_n there is a zero of P_{n+1}.

Hint: Prove by induction that $P'_{n+1} P_n > P_{n+1} P'_n$ and consider the signs of P_{n+1} and P'_n for $P_n(x) = 0$.

The content of examples X 01, X 05, X 06 is closely connected to the theory of passive electrical network synthesis. X 07 is an example of a rather general property of orthogonal polynomials, namely the interlacing property of their zeros. Here the Legendre polynomials are treated.

X 08 • **I 20 I 66**

Let f be a positive continuous function on the interval [0, 1]. Prove that

$$\left(\int_0^1 \frac{1}{f(x)}dx\right)^{-1} \leq e^{\int_0^1 \log f(x)dx} \leq \int_0^1 f(x)dx$$

provided that the integrals exist.

Hint: Consider definition of the integral by integral sums and inequalities between harmonic, geometric and arithmetic means. Or alternatively, apply the inequality $\int_0^1 \Phi(f(x))dx \leq \Phi(\int_0^1 f(x)dx)$ for any concave function Φ. This inequality is attributed to Jensen.

X 09 • **I 23 I 66** **↑ X 08**

Let a sequence $\{a_n\}$ be defined by $a_{n+1} - a_n = d$ with $a_1, d > 0$. Define

$$G_n = (a_1 a_2 \cdots a_n)^{\frac{1}{n}},$$

$$A_n = (a_1 + a_2 + \cdots + a_n)/n$$

and prove that

$$\lim_{n\to\infty} \frac{A_n}{G_n} = \frac{e}{2}.$$

Hint: Put $c = \frac{a_1}{d}$ and write $\log \frac{G_n}{A_n}$ as an integral sum of $\int_0^1 \log f(x)dx$. Alternatively, use the Euler function Γ.

X 10 ••

For a polynomial with complex zeros z_i, $i = 1, 2, \ldots, n$ all zeros of its derivative belong to the convex hull of z_i. Prove this statement.

Hint: Consider the logarithmic derivative of P and take into account that

$$\frac{1}{z-a} = \frac{\overline{z}-\overline{a}}{|z-a|^2}.$$

This statement is known as the Lucas Gauss theorem. It is illustrated at http://demonstrations.wolfram.com/LucasGaussTheorem/

X 11 • ↓ **X 12**

Prove that the roots of any of the polynomials $p(x) = \int_a^x (t-b)^2 dx$ are vertices of such a (nondegenerated) triangle that b is the center of its inscribed circle.

X 12 •• ↓ **X 15**

Consider the cubic polynomial $p(x)$ with zeros at -2λ and $\lambda \pm i\mu$, $\lambda, \mu \in \mathbf{R}$. Prove that there exists an ellipse with its foci at the zeros of $p'(x)$ such that it is inscribed in the triangle formed by three zeros of $p(x)$ with tangent points which halve the sides of this triangle.

Hint: Consider the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and its tangent line at the point (x_1, y_1) in the form $b^2x_1x + a^2y_1y = a^2b^2$.

X 13 •• [M] ↑ **X 12**

Consider a triangle with vertices $(-1, 0), (1, 0), (p, q)$. Find the equation of a (unique) ellipse inscribed in this triangle in such way that the midpoints of the sides of the triangle are the tangent points of the ellipse with the sides of the triangle.

Such an ellipse was investigated by Jacob Steiner (see http://mathworld.wolfram.com/SteinerInellipse.html). At this web page many other interesting properties of this ellipse can be found. In particular, it has the property shown in 1979 that it is the ellipse of maximal area inscribed in a triangle.

10

X 14 ••• [M] **PX 14 V 19** ↑ **X 13**

Find a proof of Marden's theorem which reads as follows: Let $p(z)$ be a cubic polynomial with complex coefficients, and whose roots z_1, z_2, z_3 are noncollinear points in the complex plane. Let T be the triangle with vertices at z_1, z_2 and z_3. There is a unique ellipse inscribed in T and tangent to the sides at their midpoints. The foci of this ellipse are the roots of $p'(z)$.

X 11 and X 12 are special cases of Marden's theorem. For the proof of Marden's theorem see http://www.american.edu/cas/mathstat/People/kalman/pdffiles/mardenAMM.pdf, where the rather long history and various methods in proving the theorem and also further references are presented. Demonstration can be found at http://demonstrations.wolfram.com/MardensTheorem/

X 15 •• **I 63 I 64** ↓ **X 16**

Let f be a continuous one-to-one function on $\mathbf{R}$. Consider a pair of sequences $(\{a_n\}, \{b_n\})$ defined as follows:

$$a_{n+1} = f(tf^{-1}(a_n) + (1-t)f^{-1}(b_n)),$$
$$b_{n+1} = f(sf^{-1}(a_n) + (1-s)f^{-1}(b_n))$$

where f^{-1} is the function inverse to f and $s, t \in [0, 1]$ are fixed. Prove that the limits of a_n, b_n exist and that they are equal. Find this limit for given a_0, b_0.

Hint: Consider the sequences $\alpha_n = f^{-1}(a_n)$, $\beta_n = f^{-1}(b_n)$.

X 16 •• **I 63 I 64** ↑ **X 15**

Let f, g be continuous one-to-one functions on $\mathbf{R}$. Consider a pair of sequences $(\{a_n\}, \{b_n\})$ defined as follows:

$$a_{n+1} = f(tf^{-1}(a_n) + (1-t)f^{-1}(b_n)),$$
$$b_{n+1} = g(sg^{-1}(a_n) + (1-s)g^{-1}(b_n))$$

where f^{-1} and g^{-1} are functions inverse to f and g, respectively and $s, t \in [0, 1]$ are fixed. Prove that the limits of a_n, b_n exist and that $\lim a_n = \lim b_n$.

Hint: Consider the sequences $\alpha_n = f^{-1}(a_n)$, $\beta_n = g^{-1}(b_n)$ and the function $h = g^{-1} \circ f$. Write the conditions using α_n, β_n and h.

The number $\mu(a, b) = f(tf^{-1}(a) + (1-t)f^{-1}(b))$, with $0 \le t \le 1$ defines a generalization of various mean values.
For $t = \frac{1}{2}$ and $f(x) = x$ we obtain the arithmetic mean, for $f(x) = \frac{1}{x}$ the harmonic mean, for $f(x) = \log x$ the geometric mean.
With $t \neq \frac{1}{2}$ these values are known as weighted means.
The limits for two different means mentioned above and with any suitable pair of functions f, g are not known.

X 17 • **I 29**

For a given integer N let $0 = x_0 < x_1 < \cdots < x_N = 1$. Prove that if the ratios of each segment $x_{k+1} - x_k$ to the previous one are constant, say λ, then $\lambda = \frac{1}{x_1}$ and $\lambda^N = 1 + \lambda + \cdots + \lambda^{N-1}$. Prove also that for $N \to \infty$ there is $x_1 = \frac{1}{2}$. (We formally assume that the segment previous to $x_1 - x_0$ has length equal to 1.)

Hint: Note that for $N = 2$ the value of λ is the golden segment ratio.

Supplementary Material

Definitions

A set $E \in \mathbf{C}^n$ is called convex if $\lambda x + (1-\lambda)y \in E$ whenever $x \in E$, $y \in E$ and $0 \le \lambda \le 1$.
Given n points $z_k \in \mathbf{C}$, the set of all $z = \lambda_1 z_1 + \lambda_2 z_2 + \cdots + \lambda_n z_n$ with $\lambda_k \ge 0$ for all $k = 1, 2, \ldots, n$ and $\sum_{k=1}^{n} \lambda_k = 1$ is called the convex hull of $\{z_k\}$.

Consider a function $f : I \to \mathbf{R}$ where $I = [a, b]$ is a finite interval, and an n-tuple of points x_k, $a < x_1 < x_2 < \cdots < x_n < b$ with $\Delta x_k = x_{k+1} - x_k$, $\lambda = \max_k \Delta x_k$. Chose ξ_k such that $x_k \le \xi_k \le x_{k+1}$ and construct the sum

$$\sigma = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k.$$

If $\lim_{\lambda \to 0} \sigma$ exists (for any choice of ξ_k) then this limit is called the definite integral of f on the interval $[a, b]$. It is denoted as $\int_a^b f(x)dx$.

10

The Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

for all complex x such that $\mathrm{Re} x > 0$ or alternatively by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!(n+1)n^z}{z(z+1)\cdots(z+n)}.$$

$\Gamma(n+1) = n!$ for all positive integers n.

Plans of Solution

PX 14

1. Justify that it is no loss of generality to assume that the vertices of the triangle are chosen as $(-1,0)$, $(1,0)$, (p,q). Hence, the ellipse of X 13 can be used.
2. Show that the center of the ellipse is at the center of gravity of the triangle, i.e. at the point $(p/3, q/3)$.
3. By shifting the origin of the coordinate system to the point $(p/3, q/3)$ find the equation of the ellipse as
$$q^2\xi^2 - 2pq\xi\eta + (3+p^2)\eta^2 = q^2/3.$$
4. Write the equation of the ellipse in matrix form
$$(\xi,\eta)B\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \frac{q^2}{3}$$
with
$$B = \begin{pmatrix} q^2 & -pq \\ -pq & 3+p^2\end{pmatrix}.$$
Its eigenvectors ν_1, ν_2 determine the direction of the two axes of the ellipse. Let λ_1, λ_2 be the eigenvalues of B. The lengths of the half-axes are given by $\frac{q}{\sqrt{3\lambda_1}}$ and $\frac{q}{\sqrt{3\lambda_2}}$, since
$$\frac{q^2}{3} = \nu_1 B \nu_1^T = \lambda_1|\nu_1|^2$$
and similarly for ν_2. The foci are at the distance $\sqrt{\frac{q^2}{3\lambda_1} - \frac{q^2}{3\lambda_2}}$ from the origin, where λ_1 is the smaller of the eigenvalues.
(Eigenvalues[B], Eigenvectors[B] are the corresponding commands in MATHEMATICA®, while in Maple® the command Eigenvectors is contained in the Linear Algebra Package.)
5. With the known direction and distance of the foci from the origin, find their coordinates.
6. Shift the ellipse back to its center at $(p/3, q/3)$ and find the 'shifted' foci. Construct the normalized quadratic polynomial with these two foci as its roots. This polynomial in the z variable reads as follows: $-1 - 2pz - 2iqz + 3z^2$.
7. Identify this polynomial with the derivative of the polynomial with roots at the points $-1, 1, p+iq$ in the complex plane.
8. A side effect of these calculations is that the area A of the inscribed ellipse is $A = \frac{\pi q}{3\sqrt{3}}$.

Further References

Aczél J., Functional Equations: History, Applications and Theory, Kluwer, Dordrecht, 2002.

Marden M., Geometry of Polynomials, AMS, Providence, 1966.

Part IV
Appendix

11 Answers to Problems

Mappings, Composite and Inverse Functions (M)

M 01

The set of x, where f_{-1} is the inverse mapping of f should be specified.

M 02

(i) and (ii) follows from the definition of increasing functions; in (iii) the sum $f+g$ is always increasing, but for $f(x) = -\exp(-2x)$ and $g(x) = -\exp(-x)$ neither fg nor f/g is increasing on $\mathbf{R}$.

M 03

The function need not be continuous. The function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ x+1 & \text{if } x \text{ is irrational} \end{cases}$$

is nowhere continuous, nevertheless it has an inverse.

M 04

For $0 \le x \le 2$ the inverse f_{-1} of f exists and $f_{-1} = f$.

M 05

$f'_{-1}(y) = \frac{1}{f'(f_{-1}(y))}$.

J. Gregor, J. Tišer, *Discovering Mathematics*,
DOI 10.1007/978-0-85729-064-9_12, © Springer-Verlag London Limited 2011

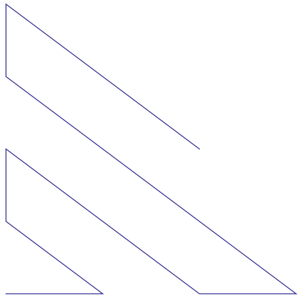
11

M 06

There are infinitely many such mappings (in fact uncountably many) One such mapping is $f(m,n) = (2m-1)2^{n-1}$. Its inverse is given by the following algorithm: Let $f(m,n) = k \in \mathbf{N}$.

1. Divide k by 2 repeatedly untill the result is an integer. Put $n = 1 +$ number of divisions.
2. The remainder is an odd integer; represent it as $2m-1$. Then $k = f(m,n)$.

Another such mapping can be derived from a meander passing the integer points as follows (starting at the point (1, 1)).



The mapping f now satisfies the equation

$$f(m-1, n+1) = (-1)^{m+n+1} + f(m,n)$$

with the following boundary conditions:

$$f(2m, 1) = 1 + m(2m-1), \qquad f(1, 2n-1) = 1 + (2n-1)(n-1).$$

Its solution is

$$f(m,n) = (m+n-1)(m+n-2)/2 + q,$$

where $q = m$ or n for $m+n$ even or odd, respectively. Note that $f(m,n) - q$ is the sum of positive integers up to $m+n-2$. Hence the inverse mapping is given by the following algorithm.

For $k = f(m,n)$ subtract successively $1, 2, 3, \ldots$ untill the result becomes nonpositive. Then the last subtracted number equals $m+n-1$ and the last positive result equals q. E.g. for $k = 24$ we obtain $23, 21, 18, 14, 9, 3, -4$. Hence $m+n-1 = 7$ and $q = 3$; since $m+n$ is even, we have $q = m$ and $24 = f(3,5)$. Similarly, $35 = f(2,7)$.

M 07

$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$, where the first summand is even and the second one is odd. In particular, $\exp x = \cosh x + \sinh x$ and $\exp(\mathbf{i}x) = \cos x + \mathbf{i} \sin x$.

$$\cosh^2 x - \sinh^2 x = 1,$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$

$$2\cosh^2(x/2) = \cosh x + 1,$$

$$2\sinh^2(x/2) = \cosh x - 1, \quad \text{etc.}$$

M 08

$A = \arg\sinh x$, and therefore $x = \sinh A$.

M 09

Put $f(0) = 1$ and calculate $f(x) - f(-x)$.

M 10

1. Is simple.
2. If $\mathrm{Ev}(f) \equiv 0$, then the implication is evidently true. Otherwise it need not be true although the inverse f_{-1} may exist. E.g. $f(x) = x + 1$, $f(x) = (x + 1)^3$.

M 11

Denoting $R(f)$, $D(f)$ the range and domain of f, respectively, then $R \subset D$.
Examples:

$$1/x \quad \text{for } x > 0 \text{ or } x < 0,$$

$$\frac{x+1}{x-1} \quad \text{for } x > 1 \text{ or } x < 1,$$

$$\sqrt{a^2 - x^2} \quad \text{for } 0 < x < a,$$

$$-\sqrt{a^2 - x^2} \quad \text{for } -a < x < 0,$$

$$(-x + \sqrt{8}\sqrt{3 - x^2})/3 \quad \text{for } 1/\sqrt{3} < x < \sqrt{3},$$

$$(1 + x\sqrt{2} - \sqrt{1 + 4x\sqrt{2}})/\sqrt{2} \quad \text{for } x \in \left[\frac{-1}{4\sqrt{2}}, \frac{3}{4\sqrt{2}}\right], \quad \text{etc.}$$

If f is an even continuous function and the equation

$$\frac{x-y}{\sqrt{2}} = f\left(\frac{x+y}{\sqrt{2}}\right)$$

has a solution $y = \varphi(x)$ which is monotonic on an interval T, then φ is a function inverse on T to itself.

11

M 12

The zeros are at $x_k = \cos\frac{(2k+1)\pi}{2n}$ with $k = 0, 1, \ldots, n-1$, hence $|x_k| < 1$ and all the zeros must be simple since their number is equal to the degree of the polynomial.

M 13

For $\frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}$ both statements remain true, i.e. the function is a polynomial of degree n and all its zeros are simple with $|x_k| < 1$.

$\sin((n+1)\arccos x)$ has all its zeros simple with $|x_k| < 1$ although it is not a polynomial.

M 14

For $-1 < x < 1$

$$\arcsin x = \arctan\frac{x}{\sqrt{1-x^2}} = \operatorname{arccot}\frac{\sqrt{1-x^2}}{x}.$$

For $x > 0$

$$\arctan x = \arcsin\frac{x}{\sqrt{1+x^2}} = \arccos\frac{1}{\sqrt{1+x^2}}.$$

M 15

$\cos(2n\arctan x) + \mathbf{i}\sin(2n\arctan x) = (\frac{1+\mathbf{i}x}{1-\mathbf{i}x})^n$, where $\mathbf{i}^2 = -1$.

The considered functions have only real and simple zeros located symmetrically with respect to the origin, they are rational and their denominator has n-tuple zeros at the points $\pm\mathbf{i}$.

M 17

$\zeta = 3x + y$, $\eta = -5x + y$.

The Jacobian of the map equals 8 and it is equal to the ratio of the areas of the image and the original parallelogram.

M 18

With area $A = (r_2 - r_1)(\alpha_2 - \alpha_1)$ the area $\Phi(A) = (r_2^2 - r_1^2)(\alpha_2 - \alpha_1)/2$ and therefore

$$\lim_{r_2\to r_1}\frac{\text{area}\,\Phi(A)}{\text{area}\,A} = r_1 = J(\Phi(r,\alpha))_{r=r_1}.$$

Notice that this ratio is independent of α.

M 19

Let u, v be orthonormal vectors in $\mathbf{R}^3$ and $a \in \mathbf{R}^3$. Then the pair $(\langle a, u\rangle, \langle a, v\rangle)$ are coordinates of the projection of the point $a \in \mathbf{R}^3$ into the plane with the coordinate system determined by u, v.

Let the equation of the plane be considered as $ax + by + cz = 0$, $a^2 + b^2 + c^2 \neq 0$. To establish a coordinate system in this plane we can use the vectors, e.g. $(b, -a, 0)$ and $(c, 0, -a)$ if $a \neq 0$.

To form an orthonormal basis we may use the Gram Schmidt orthogonalization procedure. In MATHEMATICA®:

```
<< LinearAlgebra`Orthogonalization`\\
{u, v} = Orthogonalize[{{b, -a, 0}, {c, 0, -a}},
  Method -> "GramSchmidt"]
```

In Maple®:

```
> with(LinearAlgebra);  w1:=(b, -a, 0); w2:=(c,0,-a);\\
[u,v]:= GramSchmidt([w1,w2])
```

An arbitrary point $q \in \mathbf{R}^3$ is projected into a point (ξ, η) in this plane as $(\xi, \eta) = (\langle q, u\rangle, \langle q, v\rangle)$.

M 20

1. The points on a unit sphere are described by spherical coordinates $x = \sin p \cos q$, $y = \sin p \sin q$, $z = \cos p$, and on the cylinder of radius 1 with z as its axis $x = \cos q$, $y = \sin q$, $z = t$. Then $f(q, p) = (q, t) = (q, \operatorname{ctg} p)$.
2. Consider two curves on the sphere: first $p = p(t)$, $q = q(t)$, second $p = \text{const}$, $q = t$ and calculate their angle. Under the mapping we obtain curves $x = \cos q(t)$, $y = \sin q(t)$, $z = z(t)$, where $q(t)$ remains unchanged and $z(t)$ is unknown. The condition to preserve the angles of the two curves yields an equation for the unknown $z(t)$.

   After simplification we obtain $z' = \frac{p'}{\cos p}$, hence $z = \log \tan(\frac{p}{2} + \frac{\pi}{4})$.

M 21

If (x, y) are points of the tangent plane and (ξ, η, ζ) are points of the sphere then the mapping is defined as

$$x = \frac{2\xi}{2-\zeta}, \qquad y = \frac{2\eta}{2-\zeta}.$$

Its inverse can be found with $\xi^2 + \eta^2 + (\zeta - 1)^2 = 1$ as

$$\xi = \frac{4x}{x^2+y^2+4}, \qquad \eta = \frac{4y}{x^2+y^2+4}, \qquad \zeta = \frac{2(x^2+y^2)}{x^2+y^2+4}.$$

The angle α of two rays

$$x = a_i t + x_0, \qquad y = b_i t + y_0, \qquad a_i^2 + b_i^2 = 1, \quad i = 1, 2$$

is determined by $\cos\alpha = a_1 a_2 + b_1 b_2$. On the sphere the corresponding angle of the two

circles can be obtained after substitution of the rays into (ξ, η, ζ) and differentiation with respect to t.

M 22

$$\int_0^a (f(x))^n dx \le \frac{1}{M}\frac{f^{n+1}(a)}{n+1} \quad \text{if } f' \ge m > 0.$$

M 24

$$\int_1^2 \arcsin(1/x)dx = \int_{\pi/6}^{\pi/2} \frac{1}{\sin x}dx - \pi/6.$$

M 25

$s_{-1}(x) = \arcsin x$, $ss_{-1}(x) = \arg\sinh x$.

M 28

$\mathrm{sl}(x) = \frac{1}{\sqrt{2}}(K(\frac{1}{\sqrt{2}}) - F(\arcsin\sqrt{1-x^2}, \frac{1}{\sqrt{2}}))$ where K and F are the complete and incomplete elliptic function of the first kind, respectively. $\mathrm{sl}(x)$ can also be represented in terms of the incomplete Beta function.

M 29

With $\sin\frac{\alpha}{2} = \eta\sin\frac{\alpha_0}{2}$ obtain

$$\begin{aligned} t &= \sqrt{\frac{l}{g}}\int_\eta^1 \frac{1}{\sqrt{(1-\eta^2)(1-\eta^2\sin^2\frac{\alpha_0}{2})}}d\eta \\ &= \sqrt{\frac{l}{g}}\left(K\left(\sin^2\frac{\alpha_0}{2}\right) - F\left(\arcsin\eta, \sin^2\frac{\alpha_0}{2}\right)\right). \end{aligned}$$

The initial angle is halved for $\eta = \frac{\sin\frac{\alpha_0}{4}}{\sin\frac{\alpha_0}{2}} = \frac{1}{2\cos^2\frac{\alpha_0}{4}}$.

E.g. for $\alpha_0 = \pi/2$ and $l = 1$ we obtain $\arcsin\eta = 0.571859$ and $t = 0.31933$ $(K(0.146447) - F(0.571859, 0.146447)) = 0.337649$ seconds.

For the linear model of the pendulum the answer reads $t = 0.334402$. This is obtained from the equation $\psi'' + A\psi = 0$, where $A = mgl$ and $\psi(0) = \pi/2$, $\psi'(0) = 0$.

M 31

With $b = 1$, $0 < a < 1$ the length s of the ellipse is

$$s(t) = \int_0^t \sqrt{1-(1-a^2)\sin^2 x}dx = E(t,k)$$

where $k = \sqrt{1-a^2}$ is the distance between the center and the focus of the ellipse.

M 32

$K(1-\frac{a^2}{b^2})/b$, where K is obtained by the EllipticK command.

M 33

The hint yields

$$a_{n+1} < a_n, \qquad b_{n+1} > b_n, \qquad a_{n+1} > b_n, \qquad b_{n+1} < a_n.$$

Hence the sequences are monotonic and bounded. Their limits, say A, B, exist and $A = (A+B)/2$ implies that they must be equal.

M 34

Use the results of M 30.

$$\lim a_n = \lim b_n = \frac{\pi}{2\int_0^{\pi/2}\sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt} = \frac{\pi a}{2K(1-\frac{b^2}{a^2})} \quad \text{for } a^2 > b^2,$$

where K is the complete elliptic integral of the first kind.

M 35

$f(1) = f(-1/2 + \mathbf{i}\sqrt{3}/2)$

M 38

The quadratic polynomial

$$p(x)2.98633 - 2.04845x - 0.266435x^2$$

interpolates the function inverse to $(\sin x)/x$ at points $(\sin x_k)/x_k$, assuming the values $x_k = 1.9 \pm k0.05$, $k = -1, 0, 1$. Hence, the value $p(0.5) = 1.89549$ is an approximation of the solution of $\sin x = x/2$.

M 39

Composition corresponds to the multiplication of the matrices (a_{ik}) and (b_{ik}).

M 40

Since the mapping under consideration is linear it is sufficient to check that $\xi^2 + \eta^2 = x^2 + y^2$. It follows that the mapping with $A = (a_{ik})$, $i, k = 1, 2$ is distance-preserving iff there exists an angle φ such that

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

The mapping defines rotation of the coordinate system with the origin as its center.

M 41

f is angle preserving for all $a \neq 0$ and it is distance-preserving for $|a| = 1$.

11

M 43

$f(z) = az + b$, $|a| = 1$.

M 50

$F(x) = \frac{1}{2}\frac{x^3-x^2-1}{x^2-x}$ is the only solution.

M 51

1. Examples are: $F(x) = f(x - 1/2)$, where f is an arbitrary even function, and further, with any function G we find that $F(x) = G(f(x - 1/2))$ is also a solution.
2. Similarly with $a/2$ instead of $1/2$.
3. Examples are: $H(x) = f(\frac{1}{2}(x + 1/x))$, where f is an arbitrary function, but this solution does not give all the required functions.

The graphs of F and G are symmetrical about the straight lines $x = 1/2$ and $x = a/2$, respectively.

M 52

$F(0) = 1$, if $F(x)$ satisfies the equation then so does $F(ax)$ for all $a \neq 0$. The coefficients a_k of the power series satisfy the equation $a_{m+n}\binom{m+n}{n} = a_m a_n$ and therefore

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Infinite Sequences (I)

I 01

For (iii) and (vi) the sequences diverge. All others have a limit equal to 1.

```
Limit[( Sqrt[n + 1] - Sqrt[n - 1]) Sqrt[n]  , n -> Infinity]
ListPlot[Table[(1 + (-1)^n/n)^n, {n, 1, 30}]]
```

I 02

(i) $x_n = nd + x_0$, diverges to $\mathrm{sign}(d)\infty$.
(ii) $x_n = x_0 + \frac{1}{2}((-1)^n - 1)$, diverges.
(iii) $x_n = \lambda^n x_0$, if $x_0 = 0$ converges, if $x_0 \neq 0$ converges for $-1 < \lambda \leq 1$, otherwise diverges.
(iv) $x_n = n!$, diverges.
(v) $x_n = x_0^q$, $\frac{1}{q} = 2^n$, converges.
(vi) $x_n = \alpha^{\frac{n(n+1)}{2}}$, if $-1 < \alpha \leq 1$ it converges, otherwise diverges, the identity $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ was used.
(vii) $x_n = \exp(\exp(\cdots\exp(0)\cdots)$, the function exp is applied n times, diverges.

I 03

0, does not exist.

I 04

Try $A_n = na_n$, $B_n = nb_n$.

I 05

Use Theorem D (Squeeze theorem).

I 06

$\lim x_n = \lim y_n = \lim z_n = 0$, w_n is convergent iff c_n is convergent.

I 07

(i) Either $\lim f_n = 0$ or $\lim g_n = 0$.
(ii) If f_n is convergent then $\lim f_n = 0$.
(iii) Yes, see I 04.

I 09

$\lim z_n = \lim w_n = A$.

I 10

(i) The area of an isosceles triangle with the central angle $2\pi/n$, and with the side and altitude respectively of length 1, equals $1/2\sin(2\pi/n)$, $\tan(\pi/n)$, respectively. Hence $A(n) = n\tan(\pi/n)$, $a(n) = \frac{1}{2}n\sin(2\pi/n)$.
(ii) Use $\sin t = 2\sin(t/2)\cos(t/2)$ and $\tan(t/2) = \sin t/(1+\cos t)$.
(iii) $k = 5$.

I 11

For (i), $N > \frac{2}{\epsilon} + 1$ is sufficient, the same result holds true for $n^{\frac{1}{n}}$.
For (ii), we obtain $n > 1/\sqrt{6\epsilon}$, use that $\sin x \leq x - \frac{1}{6}x^3$ for x small.
For (iv), $N > \frac{(1+\epsilon)^2}{(1+\epsilon)^2-1}$ is sufficient.
The main distinction is in the power of ϵ in these estimates.

I 13

In I 01 $O(n^{-1})$, $O(n^{-2})$, $O(n^{-1})$, $O(n/\log n)$, for (i), (ii), (iv) and (v), respectively.
In I 10 both are $O(n^{-2})$.
In I 13 (i) $p = 1$, (ii) $p = 2$.

11

I 14

(i) $O(n-p)$, (ii) $O(n^{-\min(p,q)})$, (iii) $O(n-q-p)$.

I 15

(i) $O(n-p)$, (ii) $O(n^{-2p)})$, (iii) $O(\mu^{-p}(n))$, since $|w(n)| = |x(\mu(n))| \le C|\mu^{-p}(n)|$.

I 16

(ii) Put $\mu(n) = \min\{m \ge n : |x_m - A| \le g_n\}$ and $y_n = x_{\mu(n)}$. (i) is a special case of (ii) with $g_n = n^{-p}$.

I 17

$$x_n = O(\lambda^{-n}).$$

I 20

$\log 2$, use $\int_{n+1}^{2n+1} \frac{1}{x} dx \le a_n \le \int_n^{2n} \frac{1}{x} dx$.

I 21

$\lim y_n = \max\{a_1, a_2, \dots, a_N\}$, $\lim z_n = \sup\{a_1, a_2, \dots\}$.

I 23

$$\lim y_n = 1.$$

I 24

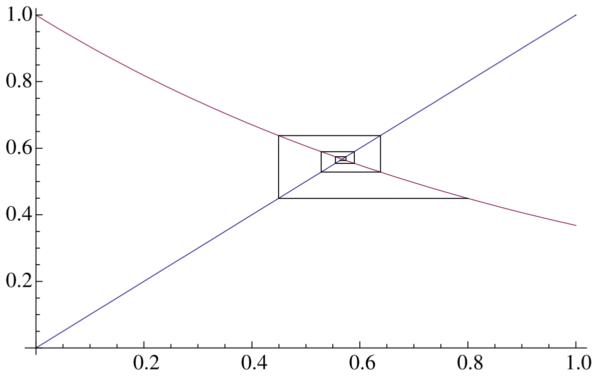
$$\lim x_n = \max_{x \in [a,b]} |f(x)|.$$

I 25

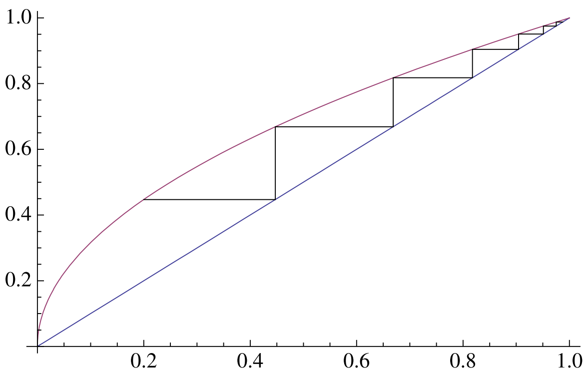
1. $\frac{x_{n+1}}{x_n} = (\frac{1+1/(n+1)}{1+1/n})^{n+1} \frac{n+1}{n} = (1 - \frac{1}{(n+1)^2})^{n+1} \frac{n+1}{n} \ge (1 - \frac{1}{n+1}) \frac{n+1}{n} = 1.$
2. $\frac{y_n}{y_{n+1}} = (\frac{1+1/n}{1+1/(n+1)})^{n+2} \frac{n}{n+1} = (1 + \frac{1}{n(n+2)})^{n+2} \frac{n}{n+1} \ge (1 + \frac{1}{n}) \frac{n}{n+1} = 1.$

I 26

For $\exp(-x)$ with $x_0 = 0.8$ we obtain



and for $\sqrt{x}$ with $x_0 = 0.2$ we obtain



I 27

If $x_0 > 0$ then $\lim x_n = \sqrt{a}$.

If $x_0 < 0$ then $\lim x_n = -\sqrt{a}$.

$x_n = O(2^{-n})$.

I 28

In MATHEMATICA®

```
Solve[2 x ==1+x^3,x]
```

In Maple®

```
fsolve(2*x = 1+x^3, x)
```

$$\lambda_1 = -(1+\sqrt{5})/2, \qquad \lambda_2 = (\sqrt{5}-1)/2, \qquad \lambda_3 = 1,$$

$$\lim x_n = \begin{cases} -\infty & \text{for } x_0 < \lambda_1 \\ \lambda_2 & \text{for } \lambda_1 < x_0 < \lambda_3 \\ \infty & \text{for } x_0 > \lambda_3. \end{cases}$$

11

I 33

(ii) $\lim \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$, (v) $\lambda_{1,2} = (1 \pm \sqrt{5})/2$, (vi) $F_n = \frac{1}{\sqrt{5}}\lambda_1^n - \frac{1}{\sqrt{5}}\lambda_2^n$.

I 34

The limit exists and equals 1.

I 35

$$\lim z_n = \begin{cases} 0 & \text{for } |a| < 1 \\ \infty & \text{for } |a| > 1, \end{cases}$$

for $|a| = 1$, $a \neq 1$ the limit does not exist.

I 36

Since $|Ax| \leq |A||x|$ it is sufficient that the moduli of all eigenvalues are less than 1.

I 43

1. $B(n) = B(\lfloor n/2 \rfloor) + 1$, $B(1) = 1$
2. In MATHEMATICA®:

```
Bn[k_] := Table[1 + Floor[Log[2, n]], {n, 1, k}]
```
In Maple®:
```
 F := k --> [seq(1+floor(log[2](n)), n = 1 .. k)]
```
And also
```
 Table[Floor[Log[2,n]] ,{n,20,40,5}]
Table[Length[IntegerDigits[n,2]],{n,20,40,5}]
```
or in Maple®
```
[seq(floor(log[2](n)), n = [seq(20 + 5 i, i = 0 .. 14)])]
[seq(nops(Reverse(convert(|n|, base, 2))),
n = [seq(20 + 5 i, i = 0 .. 14)])]
```
$A_k = k$.

I 44

In MATHEMATICA®:
```
 Eigenvalues[{{0.9,0.2},{-0.1, 0.6}}]
```
In Maple® use the context menu for calculating eigenvalues.

I 51

For $a_n : \{0, 1, -1\}$, for $b_n : \{0, \pm\frac{\sqrt{3}}{2}\}$, for $c_n : \{4, -2, -1\}$.

I 53

Projection of the sequence k_n from Example I 52 onto the x-axis gives the sequence a_n.

I 54

In both cases the answer is 'yes'. For the second question assume e.g. the sequence $a_1, 1, a_1, 2, a_1, 3, a_1, \ldots$.

I 61

(i) $\lim x_n = (a + 2b)/3$, (ii) $\lim x_n = \sqrt[3]{ab^2}$, (iii) $\lim x_n = \frac{3ab}{b+2a}$.

I 62

Since $x_{n+1}y_{n+1} = \cdots = x_0 y_0$, $\lim x_n = \lim y_n = \sqrt{ab}$.

I 63

$$\lim x_n = \lim y_n = \frac{\pi a}{2K(1 - \frac{b^2}{a^2})} \quad \text{for } a > b$$

where K is the complete elliptic integral of the first kind.

I 64

$$\lim x_n = \lim y_n = \frac{2a}{\pi} K\left(1 - \frac{a^2}{b^2}\right) \quad \text{for } a < b$$

e.g. for $a = 0.5$, $b = 1$ there is $\lim x_n = 0.68644\ldots$.

I 65

$\lim x_n = \lim y_n = \lim z_n$, for $x_0 = 2$, $y_0 = 3$, $z_0 = 4$ there is $\lim x_n = 2.883037\ldots$.

Periodicity (P)

P 01

(a) $\frac{2\pi}{\omega}$, (b) $\frac{2\pi\mathbf{i}}{\lambda}$, (c) m, (d) $\frac{1}{\lambda}$, (e) 1.

P 02

A sum of a rational and an irrational number is irrational. A sum of two irrationals may be rational.

11

P 03

All functions are periodic.

P 04

Clearly $\int_0^T f(x)dx = \int_{(k-1)T}^{kT} f(x)dx$. Let $a \in ((k-1)T, kT]$ for some $k \in \mathbf{Z}$. Then

$$\int_a^{a+T} f(x)dx = \int_a^{kT} f(x)dx + \int_{kT}^{a+T} f(x)dx.$$

Substitute $x = y + T$ in the last integral and obtain the result.

P 05

The functions are periodic iff $\int_0^T f(t)dt = 0$ and $\int_0^T f(t)g(t)dt = 0$, respectively.

P 06

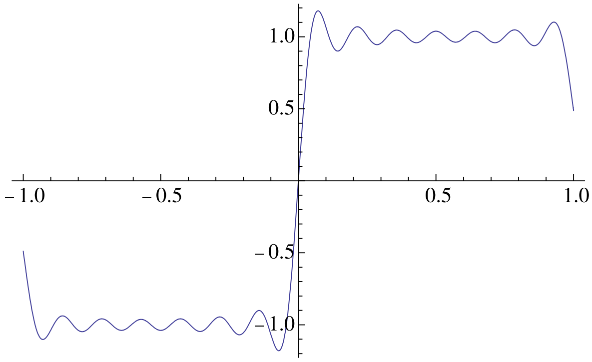
20π.

P 07

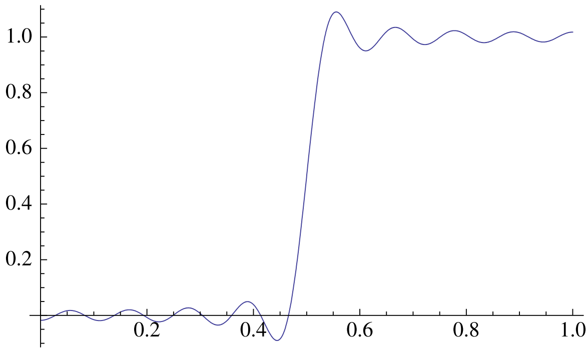
For $p, q \in \mathbf{Z}$ and $T_1/T_2 = p/q$ the period of the sum is $pT_1 = qT_2$.

P 09

For the first example we obtain (for $n = 28$)



and for the second one (for $n = 18$)



The corresponding commands in MATHEMATICA® are:

```
b[n_]=Integrate[Sign[x] Sin[n \[Pi] x/2]/2,{x,-1,1}];\\
Plot[Sum[b[n]Sin[n \[Pi] x/2],{n,1,28}],{x,-1,1}]
```

and similarly for the second case.
In Maple®:

```
b:=n->int((1/2)*signum(x)*sin((1/2)*n*Pi*x),x=-1..1);
plot(sum(b(n)*sin((1/2)*n*Pi*x),n = 1..20),x=-1..1)
```

For the first example $\max_{|x|<\delta} |f(x) - s_n(x)|$ is known to be ≥ 0.089490 for all n, although $\lim_{n\to\infty} |f(x) - s_n(x)| = 0$ for $|x - 0.5| < \delta$.

P 11

(a), (c), (d) are even functions and (b), (e), (f) are odd functions. Moreover, for (c), (e) the graph of the function in the first half of the period is reversed with respect to the x-axis in its second half. The correspondence between the groups is as follows (a) $\to$ (v), (b) $\to$ (vi), (c) $\to$ (i), (d) $\to$ (iii), (e) $\to$ (ii), (f) $\to$ (iv).

P 12

All the sequences except $a(n)$ are T-periodic.

The sequence $a(n)$ is periodic iff $\sum_{k=1}^{T} f(k) = 0$.

P 13

All the sequences are periodic with period T, where T is the least common multiple of T_1 and T_2.

P 14

$\mathrm{Re}\, a = 0$.

P 15

The characteristic equation must have simple pure imaginary zeros only and their ratio must be rational.

11

P 16

The characteristic equation must have simple pure imaginary zeros only and the ratio of any two of them must be rational. In the case of real coefficients this condition can be satisfied only if the order of the equation is even.

P 17

A periodic solution exists iff the characteristic polynomial $P(\lambda)$ satisfies the condition $P(\mathbf{i}\,T) \neq 0$. In other words: None of the summands of the right-hand side is a solution of the homogeneous equation.

P 18

$$a_{11} + a_{22} = 0, \qquad a_{11}a_{22} - a_{12}a_{21} > 0.$$

P 19

$\alpha + \bar{\alpha} = 0$ (here $\bar{\alpha} = \text{Conjugate}(\alpha)$).

P 20

This example shows the dynamics of behavior of a dynamic system

$$z' = f(z) \quad \text{with } f(0) = 0,\ f'(0) = \mathbf{i}\,.$$

The system has a periodic solution for some initial conditions and nonperiodic solutions for other initial conitions. Analyse and experiment with the following command
in MATHEMATICA®

```
Manipulate[Module[{},ClearAll[x,y];
{x[t_],y[t_]}={x[t],y[t]}/.NDSolve[
{(x')[t]==E^-y[t]Cos[x[t]]-1,(y')[t]==E^-y[t] Sin[x[t]],
x[0]== x0,y[0]== y0},{x[t],y[t]},{t,0,20}][[1]];
ParametricPlot[{x[t],y[t]},{t,0,20},
PlotRange->{{-5,2},{-4.5,2}}]],{x0,0.01,1,0.05},
{y0,0.01,1,0.05}]
```

For Maple® see the Maple® worksheet.

P 21

$a = -e^{\mathbf{i}\pi\alpha}$ for some rational value α.

P 22

$a = 0$, $b = e^{\mathbf{i}\pi\alpha}$ for some rational value α.

P 23

The characteristic equation must have only simple zeros of the form $e^{\mathbf{i}\pi\alpha}$ with rational values α. In the case of real coefficients this condition can be satisfied only if the order of the equation is even.

P 24

A periodic solution exists iff the characteristic polynomial $P(\lambda)$ satisfies the condition $P(e^{\mathbf{i}\pi/T}) = 0$. Otherwise: None of the summands of the right hand side is a solution of the homogeneous equation.

P 26

$$g(x) = \frac{b+ax}{-a+cx}, \qquad f(x) = \frac{acx - (a^2 + ad + d^2)}{c(cx+d)}.$$

P 27

$$f(x) = \frac{x\cos\frac{2\pi}{k} + \sin\frac{2\pi}{k}}{-x\sin\frac{2\pi}{k} + \cos\frac{2\pi}{k}}.$$

P 28

(b) There are infinitely many such matrices, e.g. $\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}$;
(c) AU where U is any regular integer matrix.

P 29

The matrix $A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, is a periodicity matrix of g.

P 30

All the functions are periodic with periodicity matrix A.

P 31

$AU = B$ or $A = BU$ for some matrix U with rational components.

P 32

$|\det A|$.
The columns of matrix A give coordinates of two vertices of a parallelogram with a third vertex at the origin. This parallelogram is the repeated pattern of the periodic function.

P 34

Condition is satisfied iff (x, y) is a convex combination of the vertices of the basic parallelogram.

11

P 35

For an arbitrary point (x, y) the value $f(x, y) = f(m, n)$ with $(m, n) = (x, y) - A \cdot$ Floor$[A^{-1} \cdot (x, y)]$, where Floor$[p, q]$ is the pair of largest integers less than p and q, respectively.

P 36

In MATHEMATICA® the inequalities can be solved for the unknown integers p, q as follows

```
shift[x_,y_]:=Reduce[-1 <=y-q&&y-q<1&&-2 <=y-q+Sqrt[3]
(x-pSqrt[3])
&&y-q+Sqrt[3](x-pSqrt[3])<2 &&-2 <=y-q-Sqrt[3](x-pSqrt[3])
&&y-q-Sqrt[3](x-pSqrt[3])<2,{p,q},Integers]
```

In Maple®:

```
 F:=(x, y)->Reduce(-1<=y-q and y-q<1
and -2<=y-q+sqrt(3)*(x-pSqrt(3))
and y-q+sqrt(3)*(x-pSqrt(3))<2
and -2<=y-q-sqrt(3)*(x-pSqrt(3))
and y-q-sqrt(3)*(x-pSqrt(3))<2,[p, q],SetOf(integer))
```

There is $y_0 = y - q = \pm|y \bmod 2 - 1|$ and p must be an integer satisfying the inequalities and such that $p + q$ is even. E.g. for $(0.5 + 3\sqrt{3}, 1.3)$ this command yields

```
shift[0.5 + 3Sqrt[3], 1.3]
(p==3&&q==1)||(p==3&&q==2)
```

and we conclude that $p = 3$, $q = 1$.

One of the periodicity matrices $A = \begin{pmatrix} \sqrt{3} & 2\sqrt{3} \\ 1 & 0 \end{pmatrix}$.

Finite Sums (F)

F 00

0, 0.

F 01

$$T(n) = \sum_{i\geq 0, k\geq 0}^{i+k\leq n} 1 = \frac{n(n+1)}{2}.$$

F 02

$$U(m, n) = T(n) - T(m + 1).$$

F 03

$n(n+1)/2$

`Simplify`$[n(n+1)/2+n+1-(n+1)(n+2)/2]$.

F 04

$s_n^{(1)} = an^2 + bn + c$ with $a = b = 1/2,\ c = 0$.

F 07

$s_n^{(2)} = n(n+1)(1+2n)/6,$

$s_n^{(4)} = n(n+1)(2n+1)(3n^2+3n-1)/30.$

F 08

$$n^2, \quad \frac{1}{3}n(-1+4n^2), \quad n^2(-1+2n^2), \quad \frac{1}{15}n(7-40n^2+48n^4).$$

F 09

Notice that for $p = [\sqrt{n}]$ we have $s_p \le n < s_{p+1}$, $s_p = \sum_{k=1}^{p}(2k-1)$. To achieve arbitrary precision, we should start with $100^q n$ with a suitably chosen integer q.

F 10

$$\frac{n}{2}(1+n)(2+n).$$

F 11

$s_{4n} = 4n,\ s_{4n-1} = 0,\ s_{4n-2} = -4n+1,\ s_{4n-3} = -1,$

$q_{4n} = 4n(1+4n),\ q_{4n-1} = 4n,\ q_{4n-2} = -16n^2+12n-1,$

$q_{4n-3} = -4n+3.$

F 14

Denoting by Q the matrix of the quadratic form, it is necessary and sufficient that rank $Q = 1$.

F 15

An integer has a trivial expansion iff it is a power of 2.

For the number 15 only the given expansions exist. For $n = 1050$ the number of decompositions is 6, the 'longest' starts with 30 and has 25 summands.

```
FactorInteger[2100]
{{2,2},{3,1},{5,2},{7,1}}
```

F 16

1. $\binom{m-1}{n-1}$. 2. $\binom{m+n-1}{m}$. Notice that the number of decompositions we are asking for is equal to the number of summands in $(a_1 + a_2 + \cdots + a_n)^m$.

F 18

This estimate is always better than the one in **F 17**, e.g.

```
f[x_]:={N[(Floor[Log[2,x]]+1)/2],N[Log[x+1]],
N[HarmonicNumber[x]],N[Log[x]+1],N[(Floor[Log[2,x]]+1)]}
```

F 20

$$n(n+1)H_n/2 - n(n-1)/4.$$

F 21

The Fibonacci sequence satisfies the difference equation

$$F(n+2) = F(n+1) + F(n), \quad \text{with } F(1) = F(2) = 1,$$

hence the sequence of its partial sums satisfies $s(n+3) - 2s(n+2) + s(n) = 0$ with $s(0) = 0$, $s(1) = 1$, $s(2) = 2$. The first few terms of $s(n)$ are $\{1, 2, 4, 7, 12, 20, 33, 54, 88, 143, \ldots\}$.

F 22

$$\Delta^k u(n) = \Delta^{k-1} u(n+1) - \Delta^{k-1} u(n),$$

$$\begin{aligned}\Delta^k u(n) = &\binom{k}{0} u(n+k) - \binom{k}{1} u(n+k-1) \\ &+ \binom{k}{2} u(n+k-2) - \cdots + (-1)^k \binom{k}{k} u(n).\end{aligned}$$

F 31

$$\sum_{k=1}^{n} kq^k = \frac{q(1-(n+1)q^n + nq^{n+1})}{(-1+q)^2},$$
$$\sum_{k=1}^{n} k^2q^k = \frac{q(-1-q+(n+1)^2q^n - (2n^2+2n-1)q^{n+1} + n^2q^{n+2})}{(q-1)^3}.$$

F 33

$$\sigma_n = (E-q^{n+1})(E-q)^{-1} = (E-q)^{-1}(E-q^{n+1}),$$

where E is the identity matrix with $E-q$ regular.

F 34

$$\sum_{k=0}^{n} \sin kx = \sin\frac{(n+1)x}{2}\frac{\sin(nx/2)}{\sin(x/2)},$$
$$\sum_{k=0}^{n} \cos kx = \sin\frac{(n+1)x}{2}\frac{\cos(nx/2)}{\sin(x/2)}.$$

F 35

For the first equation $x = \frac{2k\pi}{n}$ or $x = \frac{2k\pi}{n+1}$, $k \in \mathbf{Z}$. For the second $x = \frac{(2k+1)\pi}{n}$ or $x = \frac{2k\pi}{n+1}$ with $k/(n+1)$ non-integer.

F 36

$$f(x) = \frac{\sin^2(n+1)x}{\sin x}, \qquad x_0 = \pi/(n+1).$$

F 37

Take q_1 as the smallest integer such that $\frac{p}{q} - \frac{1}{q_1} \geq 0$. Conclude that $\frac{p}{q} - \frac{1}{q_1-1} < 0$ and therefore $p > pq_1 - q > 0$. As a result, the remainder $\frac{p}{q} - \frac{1}{q_1}$ will have a smaller positive numerator. Hence the number of summands must be finite.

F 38

Note that $\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}$.

F 40

The triangle with vertices $(0,0)$, (x_2, y_2), (x_3, y_3) is rectangular iff $x_2x_3 + y_2y_3 = 0$.

F 41

Let (x_1, y_1) and (x_2, y_2) be the endpoints. Assume that $x_2 \neq x_1$ and $y_2 \neq y_1$. If not, the answer is easy. Let $|x_2 - x_1|/|y_2 - y_1| = m/n$, with m, n relatively prime. Then $q = [|y_2 - y_1|/n] + 1$ or $q = [|x_2 - x_1|/m] + 1$.

F 42

$p = (|x_2 - x_1| - 1)(|y_2 - y_1| - 1) = A + 1 - B/2$, where A is the area and B is the number of points on the boundary of the rectangle.

F 43

$q = \frac{1}{2}((|x_2 - x_1| - 1)(|y_2 - y_1| - 1) - v) = A + 1 - B/2$, where v is the number of non-vertex points of the hypothenuse, A is the area and B is the number of points on the boundary of the triangle.

F 44

$p_T = A - B/2 + 1$, where A is the area, B is the number of points on the boundary. Explicitly: $p_T = \frac{1}{2}(|x_2 - x_1||y_2 + y_1| - B) + 1$, where $B = |x_2 - x_1| + y_2 + y_1 + 1 + \alpha$, and α is the number of nontrivial common divisors of $|x_2 - x_1|$ and $|y_2 - y_1|$.

F 45

$p_T = A - B/2 + 1$, where A is the area, B is the number of points on the boundary.

F 48

Comparison of the area of a circle with an inscribed square with diagonal of length $2r$ and with a circumscribed square with diagonal of length $2[\sqrt{2}r] + 2$ yields the estimate

$$r^2 + (r+1)^2 \leq p(r) \leq ([\sqrt{2}r] + 1)^2 + ([\sqrt{2}r] + 2)^2.$$

F 51

$$a_i = \frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}x^i}p(x) \quad \text{with } x = q.$$

F 52

(i) $a_0 + 10a_1 + \cdots + 10^n a_n = (a_0 + \cdots + a_n) + 9(c_1 + \cdots + 99\ldots 9c_n)$.

(ii) $a_0 + ba_1 + \cdots + b^n a_n = (a_0 + \cdots + a_n) + (b-1)(c_1 + \cdots + (b^n - 1)c_n)$. Notice that $b^n - 1$ is divisible by $b-1$. For $b+1$ we obtain

$$\begin{aligned} a_0 + ba_1 + \cdots + b^n a_n = (a_0 + a_2 + \cdots) - (a_1 + a_3 + \cdots) \\ + \big((b^2-1)a_2 + (b^4-1)a_4 + \cdots\big) \\ + \big((b+1)a_1 + (b^3+1)a_3 + \cdots\big). \end{aligned}$$

Notice that $b^{2k} - 1$ and $b^{2k+1} + 1$ are both divisible by $b+1$.

(iii) q must be odd, no such k exists.

F 53

Yes. If a_i are the binary digits in the representation of an integer n, then $b_i = (-1)^i a_i$ are the digits in the representation of n to the base -2.

F 54

1. After n years the balance equals $\sum_{k=1}^n A_k\big(1 + \frac{p}{100}\big)^{n-k+1}$,
2. $(1 + \frac{p}{36000})^{365} > (1 + \frac{p}{100})$,
3. $\frac{\mathrm{d}}{\mathrm{d}t} A_t = kA_t$, where $k = \frac{p}{100}$.

F 55

$$\left(A_0\left(1 + \frac{p}{100}\right)^q - B\right)\left(1 + \tfrac{p}{100}\right)^{n-q}.$$

F 56

Verify that $(1 + \frac{p}{100})^{70/p}$ approximately equals 2.

Inequalities (E)

E 01

$$(\sqrt{a} - \sqrt{b})^2 \geq 0.$$

E 02

$$2|a||b| \geq 0.$$

E 03

Reduce it to E 01.

E 04

$$a = b, \qquad ab = 0, \qquad a = b.$$

E 05

$|a+b| = |a| + |b|$ iff a is a positive multiple of b.
The generalization is proved by induction.

E 06

$$a^b < b^a \quad \text{if e.g. } e \le b < a \text{ or } a < b \le e.$$

E 07

For any power n there exists a value $x_0 > 0$ such that $e^x > x^n$ for all $x > x_0$. Self-evident for $n = 1$. For $n > 1$ apply the fact that $\frac{\log t}{t} \le \frac{1}{e}$ for all $t > 0$ to show that e.g. $x_0 = n^2$.

E 08

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} s_k (b_k - b_{k+1}) - s_{n-1} b_n.$$

Since $m(b_k - b_{k+1}) \le s_k(b_k - b_{k+1}) \le M(b_k - b_{k+1})$ for all $k = 1, 2, \ldots, n-1$ we obtain the result by addition with the help of $mb_n \le s_n b_n \le M b_n$.

E 09

1. Square with vertices $(\pm 1, 0)$, $(0, \pm 1)$
2. Square with vertices $(\pm 1, \pm 1)$
3. Octahedron with vertices $(0, 0, \pm 1)$, $(0, \pm 1, 0)$, $(\pm 1, 0, 0)$
4. Cube with vertices $(\pm 1, \pm 1, \pm 1)$

E 13

Alternatively, since for $0 < x < \pi/2$ the function $\frac{\sin x}{x}$ is decreasing, we have

$$\frac{\sin x}{x} \ge \frac{\sin \pi/2}{\pi/2} = \frac{2}{\pi}.$$

E 14

(a) is false, e.g. for $f(x) = x$, $g(x) = 2 - x$, $x \in [0, 1]$.
(b) is true as follows from monotonicity of integrals.

E 16

E.g. $x \geq -1$ and $p \geq 1$ is sufficient for the validity of the inequality.

E 17

The function $f(x) = (1 + x^2)e^{-x^2}$ is decreasing for all $x > 0$ and $f(0) = 1$ and therefore the inequality holds for all x.

E 18

Local extreme of $x^a - ax + a - 1$ is at $x = 1$.

E 20

If a particle moves along a straight line with initial and final velocity zero to a distance d during a time interval T then its acceleration should be greater than $(4d)/T^2$ at least at one time instant $t \in [0, T]$.

E 21

Any point B of the graph of a convex function lies below the chord passing two distinct points of the graph separated by B.

E 22

h_1 is convex;
h_2 is not convex for $f(x) = g(x) = -\sin x$, $x \in [0, \pi]$, but h_2 is convex if f and g are both positive and with derivatives of the same sign;
h_3 is not convex for $f(x) = x^2$, $x < 0$;
h_4 is not convex for $f(x) = x^2$, $g(x) = -\sin x$, $x \in [0, \pi]$.

E 23

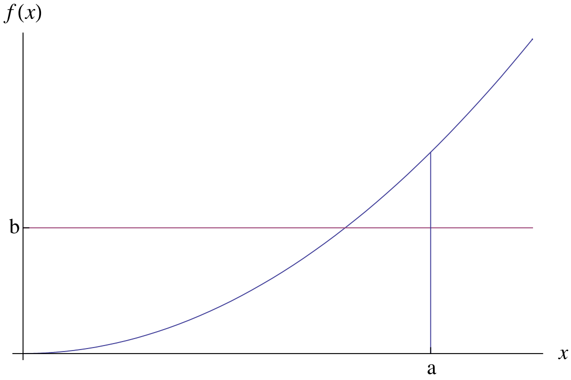
The inequality is equivalent to

$$f(a)g(b) - f(a)g(a) - \int_a^b fg'dx \leq 0.$$

The left-hand side as a function of b is decreasing on $[a, b]$.

11

E 24



In this graph find the areas corresponding to the product ab and the two integrals.

E 27

Note the convexity of $\log 1/x$ and e^x.

E 28

The set A is given by all interior and boundary points of the triangle formed by three straight lines.

$$f(x, y) \le 61, \qquad f(-5, -6) = 61, \qquad g(x, y) \le 4.4, \qquad g(1.2, 0.2) = 4.4.$$

A change of sign of the coefficient 2 of the first equation makes the triangle unbounded and neither $f(x, y)$ nor $g(x, y)$ achieves its maximum.

E 29

The best solution gives $F = 50$ with $x_1 = 5$, $x_2 = 3$.

E 30

The best solution is $F = 300$ with $x_{11} = 10$, $x_{12} = 10$.

E 35

$$\lim_{t\to 0} c_t = \lim_{t\to 0} \exp\left(\tfrac{1}{t} \log(\tfrac{1}{n} \sum x_i^t)\right) = \sqrt[n]{x_1, x_2, \ldots x_n}$$

by the l'Hospital rule.

E 36

$$4x^2 - xy + y^2 = 15/4x^2 + (y - x/2)^2,$$
$$5x^2 - 4xy + 5y^2 - 12xz - 2yz + 10z^2 = (x - 2y)^2 + (2x - 3z)^2 + (-y + z)^2.$$

E 38

Yes.

E 39

$\prec_L$ is a linear order. $(0, 0)$ is the minimal element of M.
$f(m, n) = (m + n)(m + n + 1)/2 + n + 1.$

E 40

Yes.

E 41

1. c does not exist.
2. E.g. $c = (1, 2, 3, \ldots)$.

Collocation and Least Squares Methods (Q)

Q 01

(a) Polynomials with rational coefficients.
(b) The statement is wrong: e.g. $\frac{1}{2}x^2 + \frac{1}{2}x + 1$.

Q 02

$\frac{1}{240}(570 + 83x - 6x^2 + x^3)$. The polynomial assumes rational values at rational points.

Q 03

Vandermonde's determinant is

$$V = \prod_{1 \le i < k \le n} (x_k - x_i).$$

The ordering of the nodes is irrelevant. If two of the nodes x_k are very close the matrix of the system of equations becomes ill-conditioned and practically useful results may not be obtained.

Q 04

No, only if $2x_4 \neq x_2 + x_3$, or if $p''(x_4) = \frac{p'(x_3)-p'(x_2)}{x_3-x_2}$.

Q 05

If $\sum_{k=0}^{n} \frac{f(x_k)}{\omega'(x_k)} = 0$ then Ω is of degree $\leq n-1$, e.g. the polynomial interpolating the values 3, 1, -1 at 1, 2, 3 respectively, reads $5-2x$.

Q 06

$\alpha_i = [y_0, y_1, \ldots, y_i]$, where $[y_i] = y_i$, $i = 0, 1, \ldots, n$ and

$$[y_0, y_1, \ldots, y_{i+1}] = \frac{[y_0, y_1, \ldots, y_{i-1}, y_{i+1}] - [y_0, y_1, \ldots, y_i]}{x_{i+1} - x_i}.$$

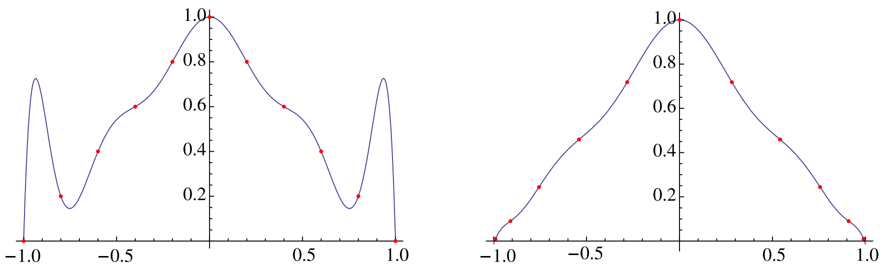
The coefficients α_i are called divided differences.

Q 07

With increasing degree of the interpolating polynomial at equidistant nodes its maximum deviation from the function f (i.e. the maximum of the error function) in this case grows. Try for example:
In MATHEMATICA®:

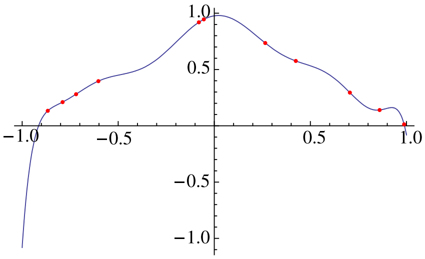
```
 f[x_] := 1 - Abs[x]
points = Table[{x, f[x]}, {x, -1, 1, Delta}];
Plot[Evaluate[InterpolatingPolynomial[points, x]],
{x, -1, 1},PlotRange -> All,
Epilog -> {Red, Map[Point, points]}] }
```

and similarly in Maple®, for various values of Delta and similarly for the other system of nodes. For n = 10 we obtain



for the equidistant and Chebyshev nodes, respectively.

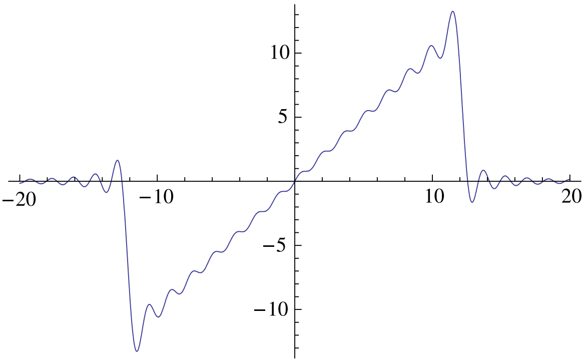
With randomly chosen nodes various results can be obtained. For one of the realizations with nodes at −0.0543388, −0.789592, 0.859975, −0.719341, 0.42382, 0.265091, −0.6034, −0.866515, −0.0804994, 0.705338, 0.987585 the interpolating polynomial appears as



Q 08

Given $x_0, x_1, \ldots, x_n$ then the function $\sum_{k=0}^{n} f(x_k) h_k(x)$ interpolates the function f at x_k, $k = 0, 1, \ldots, n$.

E.g. for $n = 15$, $\sigma = 4$ it can be obtained



Q 09

Let the matrix U with elements $u_i(x_k)$ be nonsingular. Then the matrix $A = [a_{ik}]$ of coefficients is $A = U^{-1}$.

Q 10

(2) The case $n = 1$ is clear. Take $\lambda = \max\{\lambda_i;\ i = 1, 2, \ldots, n+1\}$. Multiply the equation $\sum_{i=1}^{n+1} p_i(t) e^{\lambda_i t} = 0$ by $e^{-\lambda t}$ and consider the limit at $t \to \infty$.

(3) The solutions are successively

$$C_1 e^{-2x} + C_2 e^{-x} - \frac{1}{2} e^{-2x}(2 + 2x + x^2),$$

$$C_1 e^{-2x} + C_2 e^{-x} + \frac{1}{2} e^{-x}(2 - 2x + x^2),$$

$$C_1 e^{-2x} + C_2 e^{-x} + \frac{1}{4}(-3 + 2x),$$

$$C_1 e^{-2x} + C_2 e^{-x} + \frac{1}{36} e^{x}(-5 + 6x).$$

Q 11

$$T = \frac{1}{\alpha}\log 10, \qquad T = \frac{1}{\alpha}\log 100, \quad \text{where } \alpha = \frac{\log 2}{T_1}.$$

Q 12

At $t = 0$ the amounts of the first and second substances were 2.64113 and 10.86035, respectively.

Q 13

(a) $f(x) = ax$, $a \neq 0$.
(b) No nontrivial solution exists since $\deg(f(x)f(y)) > \deg(f(x) + f(y))$ considered as polynomials in two variables.
(c) The only solution is $f(x) = x^n$, $n \in N$, since the coefficients of the corresponding polynomial must satisfy $a_i a_j = 0$ if $i \neq j$ and $a_i^2 = a_i$.
(d) Comparing the degrees of both sides we conclude that no nontrivial solution exists.

Q 14

For $g_n(t) = a_0 + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt)$ and $G_{2n}(t) = \sum_{k=1}^{2n} u_k z^k$ we have

$$u_n = a_0, \qquad u_k = \frac{a_{n-k} + \mathbf{i}\, b_{n-k}}{2} \quad \text{for } k = 0, 1, \ldots, n-1,$$

$$u_k = \frac{a_{n-k} - \mathbf{i}\, b_{n-k}}{2} \quad \text{for } k = n+1, n+2, \ldots, 2n.$$

Q 15

$$p(x) = (2x^3 - 3x^2 + 1)p_0 + (x^3 - 2x^2 + x)m_0 + (-2x^3 + 3x^2)p_1 + (x^3 - x^2)m_1.$$

Q 16

$$\begin{aligned} p(x) = p_0&\left(1 + \frac{2(x-x_0)^3}{(-x_0+x_1)^3} - \frac{3(x-x_0)^2}{(-x_0+x_1)^2}\right) \\ &+ m_1\left(\frac{(x-x_0)^3}{(-x_0+x_1)^3} - \frac{(x-x_0)^2}{(-x_0+x_1)^2}\right) \\ &+ p_1\left(\frac{-2(x-x_0)^3}{(-x_0+x_1)^3} + \frac{3(x-x_0)^2}{(-x_0+x_1)^2}\right) \\ &+ m_0\left(\frac{(x-x_0)^3}{(-x_0+x_1)^3} - \frac{2(x-x_0)^2}{(-x_0+x_1)^2} + \frac{x-x_0}{-x_0+x_1}\right). \end{aligned}$$

.

Q 17

$$y = 1 + 2x + 5x^2/4 - x^3 \quad \text{for } 0 \le x \le 1.$$

Q 19

Assume that the shape of the cup is given by rotation of the graph of a function around the x-axis. One possible profile could be given e.g. by the function $p(x) = a + b\sqrt{x}$, $x \geq 0$ evidently $a = 3$. Its content $v(b,h)$ should be 200, which gives the equality

$$\pi \int_0^h (3 + b\sqrt{x})^2 dx = 200.$$

Using some simplification, we obtain the following system of equations: $v(b,h) = 200$, $3 + b\sqrt{h} = 5$, which yields $v(b,h) = 19h\pi$. Therefore $h = \frac{200}{19}\pi$, which leads to $h = 3.35063$, $b = 1.09261$. The 50, 100, 150 milliliter marks are at heights 1.11458, 1.95539 and 2.68717, respectively.

Q 20

For $P_n(x) = 1$ we have $y = a^{-1}$.
For $P_n(x) = 2x^2 + 3x + 1$ we obtain $y(x) = \frac{-4+11a-6a^2}{a^3} - \frac{(4-7a)x}{a^2} + \frac{2x^2}{a}$.
For given $P_n(x) = \sum_{k=0}^n b_k x^k$ and supposed $y(x) = \sum_{k=0}^n a_k x^k$ we obtain for the unknown coefficients a_i the recurrence formula

$$a_{n-k} = \frac{(n-k)!}{a}(P_n^{(n-k)}(1) - y^{(n-k+1)}(1)), \quad k = 0, 1, \ldots, n.$$

Q 21

For $y' + \alpha y = 0$ there is $a_0 = 1$ and $(k+1)a_{k+1} = -ka_k$, hence $y(x) = \sum_{k=0}^{\infty} \frac{(-\alpha x)^k}{k!}$.

Similarly, $(k+1)(k+2)a_{k+2} = \mp\alpha^2 a_k$ for $y'' + \alpha^2 y = 0$ and $y'' - \alpha^2 y = 0$, respectively, with $a_0 = 1$, $a_1 = 0$ and $a_0 = 0$, $a_1 = 1$.

In these solutions the series expansions of the functions $e^{-\alpha x}$, $\cos\alpha x$, $\sin\alpha x$, $\cosh\alpha x$, $\sinh\alpha x$ can be recognized.

Q 22

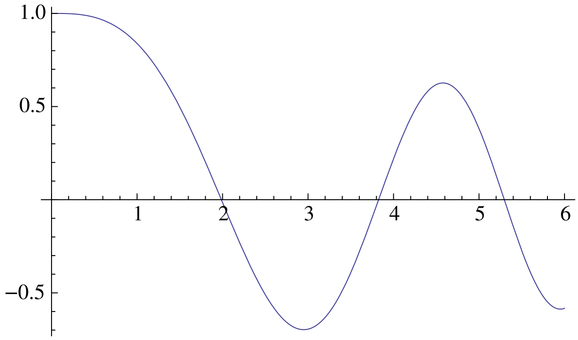
In $y(x) = \sum_{i=0}^{\infty} a_i x^i$ there are a_0 and a_1 undetermined and $a_{3k-1} = 0$.

$$a_{3k} = (-1)^k \frac{a_0}{2.3.5.6.\ldots.(3k-1)3k}, \qquad a_{3k+1} = (-1)^k \frac{a_1}{3.4.6.7.\ldots.(3k+1)3k}.$$

The series converges for all $x \in \mathbf{R}$. The solution can be expressed in terms of hypergeometric functions. For $y(0) = 1$, $y'(0) = 0$ we have

$$y(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k}}{\prod_{i=1}^k 3i(3i-1)}$$

and the graph of the function y can be obtained as



Q 23

$a_0 = 1$, $a_{2k+1} = 0$, $a_{2n} + (2n+2)a_{2n+2} = 0$, $a_{2n} = \frac{(-1)^n}{(n!)^2} 2^{-2n}$.
The series converges for all $x \in \mathbf{R}$ and gives the Bessel function $J_0(x)$.

Q 24

$$P_1 = 1 + z, \qquad P_2 = 1 + z\sqrt{2} + z^2, \qquad P_3 = (1+z)(1+z+z^2).$$

Q 25

$R[0,1](x) = \frac{1}{1-x}$, $R[1,1](x) = \frac{2+x}{2-x}$, $R[0,2](x) = \frac{2}{2-2x+x^2}$. Other $R[n,m]$ can be obtained with the command

```
PadeApproximant[ Exp[x], {x,0,{m,n}}] }
```

in MATHEMATICA® or

```
with(numapprox);pade(e^x,x,[m,n];
```

in Maple®.

Q 26

$C = (41/22, 71/22)$, $D = (43/18, 32/9)$. D coincides with the center of gravity of the triangle.

Q 27

Both (a) and (b) give the same result: $19/14$, $31/14$. The difference disappears after rewriting the equations of the lines in the form $ax + by = c$ with $a^2 + b^2 = 1$.

Q 28

Denoting $\sum t_i^2 = m_2$, $\sum t_i = m_1$, $\sum t_i x_i = r_1$, $\sum x_i = r_0$, the unknown coefficients a, b are solutions of the system $m_2 a + m_1 b = r_1$, $m_1 a + nb = r_0$ in both cases (a) and (b). The command solves the problem.

Q 29

The coefficients of the polynomial as in $P_m(t) = a_0 + a_1 t + \cdots + a_m t^m$ must satisfy the so-called normal system of equations, which reads as follows (all the sums are for $1 \leq i \leq n$):

$$a_m \sum t_i^{k+m} + a_{m-1} \sum t_i^{k+m-1} + \cdots + a_0 \sum t_i^k = \sum t_i^k x_i,$$

for $k = 0, 1, 2, \ldots, n$. This polynomial P_m for $m \geq 2$ does not minimize the sum of the squared distances of the given points and the graph of P_m.

Q 30

$x(t) = \exp(0.8105 - 0.10200t)$.

Q 31

The general solution of the differential equation is

$$y(t) = \frac{b}{1 - \exp(-bc - rt)},$$

where c is an arbitrary constant, hence $y(t) > b$. Direct application of least squares methods seems to be rather difficult due to nonlinearity of the resulting equations. Since $b < \min(y(t_k))$, put $b = 2$ and find that $\log(1 - \frac{2}{y(t_k)})$ is a linear function.

Using the least squares method (realized in MATHEMATICA® by the `Fit` command) we obtain for the given data the approximation

$$y(t) = \frac{2}{1 - \exp(-1.98194 - 0.18617t)}.$$

Q 32

$$f(x) = \frac{24}{\pi^3} x.$$

Q 33

$$q_j = \tfrac{1}{\pi} \int_0^{2\pi} f(x) \sin jx dx.$$

A conclusion similar to the formulated one can be made for any orthogonal series, i.e. for partial sums of all series of the type $\sum_{i=1}^{\infty} \lambda_n \varphi_n(x)$ such that φ_n for an orthogonal system.

Q 34

$\int_0^t T_1(x)dx$ is a trigonometric polynomial of degree n iff $a_0 = 0$.
T_1' is always a trigonometric polynomial of degree n.
$T_1 T_2$ is a trigonometric polynomial of degree $n + m$.

11

Q 36

Find that

$$\prod_{k=1}^{2} \sin \frac{t-t_k}{2} = \frac{1}{2}\left(\cos \frac{t_2 - t_1}{2} - \cos t \cos \frac{t_2 + t_1}{2} - \sin t \sin \frac{t_2 + t_1}{2}\right).$$

Q 37

$$g_k(t) = \frac{\sin \frac{t-t_0}{2} \cdots \sin \frac{t-t_{k-1}}{2} \sin \frac{t-t_{k+1}}{2} \cdots \sin \frac{t-t_{2n}}{2}}{\sin \frac{t_k-t_0}{2} \cdots \sin \frac{t_k-t_{k-1}}{2} \sin \frac{t_k-t_{k+1}}{2} \cdots \sin \frac{t_k-t_{2n}}{2}},$$

which is a trigonometric polynomial of degree n in view of Q 36.

Maximal and Minimal Values (V)

V 01

The area of a rectangle with one of its vertices at the point $(x, 0)$ is $A = 2xf(x)$ and the maximal area $A = 2af(a)$, where a is the solution of the equation $f(x) + xf'(x) = 0$. Its solution can also be found as the limit of the sequence

$$x_{n+1} = x_n - \frac{f(x_n) + x_n f'(x_n)}{2f'(x_n) + x_n f''(x_n)}$$

with $x_0 = 1/2$, say.

V 02

$f(a) = f(b)$, if $b = \frac{1}{2}(-1 - a + \sqrt{5 - 2a - 3a^2})$. The rectangle has maximal area with one of its vertices at $x = -0.200796$. This area equals 0.766981.

V 03

With α denoting the angle between the rod and one of the walls, the width $b = (d - \frac{a}{\cos\alpha}) \sin\alpha$ and we must have $b \geq (d^{\frac{2}{3}} - a^{\frac{2}{3}})^{\frac{3}{2}}$. For $d = 5$, $a = 1$ the minimal width of the corridor equals 2.66879.

V 04

With α denoting the angle between the side of the truck and the road, we obtain $d = \frac{a \sin\alpha + b \cos\alpha - w}{\sin\alpha \cos\alpha}$. For $a = 6$, $b = 5$, $w = 2$ we obtain the maximum is $d = 11.5236$.

V 05

Chose the coordinate system such that p is the x-axis. If $A = (x_1, y_1)$, $B = (x_2, y_2)$, $x_1 <$

x_2 then C will be the point $(\frac{x_2y_1+x_1y_2}{y_1+y_2}, 0)$. Since the shortest path between two points is the segment of a straight line joining them, we may find the point C on a straight line joining point A with a point B^*, symmetrical to B with respect to the straight line p.

V 06

Chose the coordinate system such that ρ is the (x, y) plane. If $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $z_1, z_2 > 0$ then $C = (\frac{z_1x_2+z_2x_1}{z_1+z_2}, \frac{z_1y_2+z_2y_1}{z_1+z_2}, 0)$. The plane containing points A, B, C has the equation

$$x(y_1 - y_2) + y(x_2 - x_1) + x_1y_2 - x_2y_1 = 0.$$

It is orthogonal to ρ.

V 07

For the points A, B the minimum of

$$d(t) = \sqrt{(\cos t - 3)^2 + (\sqrt{2}\sin t - 2)^2} + \sqrt{(\cos t + 1)^2 + \sqrt{2}(\sin t - 4)^2}$$

is at the point $t = 1.2525$, i.e. at the point $G = (0.3129, 1.3431)$ of the ellipse.

The point B must be outside of the angle of view of the ellipse taken from the point A, i.e. B must be outside of the acute angle formed by straight lines containing the point A and the points $1/11(3 \pm 2\sqrt{5}, 2 \mp 6\sqrt{5})$.

V 08

$$C = (1, 0).$$

V 09

Note that $q > 0$ implies $y > 0$ and if $|p| \geq 1 + \frac{q}{\sqrt{3}}$ then $y = 0$ and $x = \text{sign } p$, otherwise the point F is

$$F = \left(\frac{4p(3\sqrt{3} + 6q + \sqrt{3}q^2)}{3(\sqrt{3} + q)(3 + p^2 + 2\sqrt{3}q + q^2)}, \frac{3\sqrt{3} - 3\sqrt{3}p^2 + 6q + \sqrt{3}q^2}{3(3 + p^2 + 2\sqrt{3}q + q^2)}\right).$$

The following animation gives a clear picture of it.

```
 Manipulate[ListPlot[{{-1,0},{1,0},{p,q},
If[Abs[p]<1+q/Sqrt[3],{(4 p(3 Sqrt[3]+6 q+Sqrt[3] q^2))/
(3 (Sqrt[3]+q)(3+p^2+2 Sqrt[3] q+q^2)),
(3 Sqrt[3]-3 Sqrt[3] p^2+6 q+Sqrt[3] q^2)/
(3 (3+p^2+2 Sqrt[3]q+q^2))},{1,0}]},
PlotStyle->PointSize[0.02]],{p,-2,2,0.1},{q,0.2,2,0.2}]
```

For Maple® see the Maple® worksheet.

11

V 10

Put $v(y) = y'^2(x)$. Since $\frac{dv}{dx} = \frac{dv}{dy} y'$ and also $\frac{dv}{dx} = 2y'y''$ we obtain $\frac{dv}{dy} = 2y'$ and therefore $\frac{dv}{dy} + 2av = -2by$. For this linear differential equation its solution $v(y)$ yields

$$y'^2 = Ce^{-2ay} + \frac{b}{2a^2}(1 - 2ay).$$

Looking for the displacement y, where $y'' = 0$, we obtain $e^{-2ay} = -\frac{b}{2a^2}$, where the integration constant $C = \frac{be^{2ay_0}}{2a^2}(2ay_0 - 1)$, with y_0 being the initial displacement. For actual values $a = 0.01$, $b = 0.5$, $y_0 = -100$ we obtain the value $y = -45.0694$.

V 11

Let one of the straight lines be the x-axis, $A = (0, 0)$, $B = (x_2, y_2)$ (without loss of generality), $\mathbf{v}_1 = (v_{11}, 0)$, $\mathbf{v}_2 = (v_{21}, v_{22})$. Then the equations of motion are $x(t) = v_{11}t$, $y(t) = 0$ and $x(t) = x_2 - v_{21}t$, $y(t) = y_2 - v_{22}t$. We look for the minimum of $s(t) = (v_{11}t - (x_2 + v_{21}t))^2 + (y_2 + v_{22}t)^2$. Solving $\frac{ds}{dt} = 0$ for t yields

$$t = \frac{v_{11}x_2 - v_{21}x_2 - v_{22}y_2}{v_{11}^2 - 2v_{11}v_{21} + v_{21}^2 + v_{22}^2}$$

hence for s we obtain

$$s = \frac{(v_{22}x_2 + (v_{11} - v_{21})y_2)^2}{v_{11}^2 - 2v_{11}v_{21} + v_{21}^2 + v_{22}^2}.$$

V 12

With $|a|^2 = |\alpha|^2 = 1$, $\Delta = 1 - \langle a, \alpha\rangle^2$ and $\langle a, b - \beta\rangle = K$, $\langle \alpha, b - \beta\rangle = Q$, the minimal distance is

$$d^2 = \frac{1}{\Delta}(K^2 + Q^2 - 2\langle a, \alpha\rangle KQ) + |b - \beta|^2.$$

V 13

With the angles a_i we have to find the minimum of

$$F(a_1, a_2, \ldots, a_n) = \frac{d}{v}\sum_{k=1}^{n} \frac{1}{\sqrt{k}\cos a_k}$$

subject to the condition $\sum_{k=1}^{n} \tan a_k = b/d$.

With $n = 4$, $v = 1$, $b = 10$, $d = 2$ we obtain the following numerical results for a_i: $(0.476986, 0.706625, 0.919322, 1.16353)$. This result gives the coordinates of the endpoints of segments as $((0, 8), (1.03357, 6), (2.74093, 4), (5.36376, 2), (10, 0))$.

V 14

For the ellipse $x^2 + 2y^2 = 1$ and the point $(2, 3)$ the distance

$$d = \sqrt{(2 - \cos t)^2 + \left(3 - \frac{\sin t}{2}\right)^2}$$

with $t = 0.79405$, i.e. $d = 2.94535$. The shortest path from the point A to the ellipse is exactly along the normal.

V 15

The distance of the point $(1, 4)$ from $|x - y| \leq 1$ is $d = \sqrt{2}$ and the distance from the ellipse equals 1.20284. Hence this is also the distance of $(1, 4)$ from the given set.

V 16

Four solutions exist. The coordinates of the reflection points are as follows: $(0, 22/5)$, $(22/7, 0)$, $(10, 16/5)$, $(53/23, 15)$.

V 17

Numerical solution gives the reflection points $(0, 18/5)$ and $(18/7, 0)$ for one of the solutions. There exist 5 other solutions.

V 18

With the coordinate system with axes parallel to the sides of the rectangle and the origin in its midpoint, for the four points all the pairs (x_i^2, y_i^2) are equal, say $(\pm x, \pm y)$. The ellipse with minimal area has half-axes $x\sqrt{2}$, $y\sqrt{2}$ and its area is $2\pi xy$. In the example the minimal area is 20π.

V 19

The equations of the axes are $y = x(1 \pm \sqrt{2})$. The lengths of the half-axes are 1.94515 and 1.16587. The foci are at the points $(\pm 1.43852, \mp 0.595852)$.

The procedure has to find the minimum and maximum of the function $f(x, y, q) = x^2 + y^2 - q(a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_{33})$. Putting the partial derivatives equal to zero we obtain triples (x_M, y_M, q_M). The solution of the equation $\operatorname{grad} f = 0$ with q replaced by q_M gives the equations of the straight lines incident with the two axes of the ellipse, while $f(x_M, y_M, q_M)$ yields squares of the half-axes.

V 20

The statement is wrong. Its converse is true.

V 21

The height of the can $H = 2\sqrt[3]{\frac{V}{2\pi}}$ is equal to the diameter of its circular base. The surface area is $P = \sqrt[3]{54\pi V^2}$.

V 22

The points form the boundary of a square with its center at the origin and its sides $y = \pm x \mp d_0$.

V 23

For the given four points the minimal area equals 7π.

V 24

$$P_2(x) = x^2 - 1/2.$$

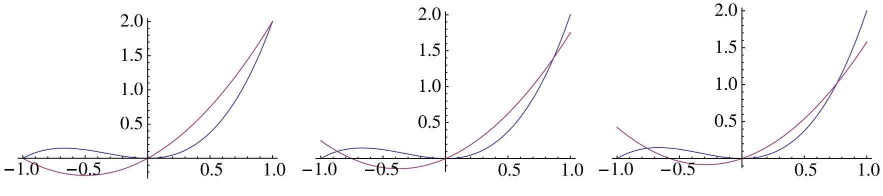
V 25

(a) $P(x) = x^3 - 3x$ or $P(x) = x^3 - \frac{3}{4}x$.
(b) $P_3(x) = x^3 - \frac{3}{4}x$.

V 26

1. $P(x) = x(1+x)$,
2. $P(x) = 3x/4 + x^2$,
3. $P(x) = 3x/5 + x^2$,
4. $P(x) = -\frac{\sqrt{2}-3\operatorname{argsinh}(1)}{4(\sqrt{2}-\operatorname{argsinh}(1))}x + x^2$.

The cases 1, 2, 4 are shown below.



V 27

$x_{\min} = 2n, n \in \mathbf{Z}$, $F(x_{\min}) = 0$, $x_{\max} = 2n+1, n \in \mathbf{Z}$, $F(x_{\max}) = 1$. F has derivatives only at all $x \in R$, $x \notin Z$.

V 29

The differences $x_n = y_{n+1} - y_n$ satisfy a difference equation and the index at which these

differences change sign is the point of the minimum. We obtain that x_n changes sign at $n = 6$ hence the minimal value of y occures at $y_7 = -10.8125$.

Arcs and Curves (A)

A 01

All the sets consist of points of circles with radius 1 except for A_7. All the circles have their centers at the origin, except for A_6, where the center is at $(1, 0)$. In A_7 with $k \neq 1$ the set is a circle with the center at

$$\left(\frac{k^2+1}{2k}\operatorname{sign}(1-k), 0\right)$$

and radius $r = 1$. For $k = 1$ the set A_7 consists of all points in the complex plane except the origin.

A 02

Eliminate t to obtain $xb_{11} - a_{11}y - a_{12}b_{11} + a_{11}b_{12} = 0$.

A 03

After elimination of t we obtain a quadratic polynomial. The corresponding determinants (see conics) are $\Delta = 0$, $\det A = -(a_{12}b_{11} - a_{11}a_{12})^4/4 < 0$.

A 04

For $n = 1$ hyperbola, for $n = 2$ straight line, for $n = 4$ a closed curve of the shape of ∞. The curve is symmetric with respect to the line $x = y$. For n even it is symmetric also with respect to the origin. With increasing n the shape in the first quadrant approaches a unit square.

A 05

$$r \geq v.$$

A 06

In general: Let the two arcs C_i have the description $(x_i(t), y_i(t))$, $t \in (a_i, b_i)$, $i = 1, 2$. Their common point exists iff the system $x_1(t) = x_2(s)$, $y_1(t) = y_2(s)$ has a solution $(t, s) \in [a_1, b_1] \times [a_2, b_2]$.

Here: Solve the system of equations $2t + 1 = 3\cos s$, $1 - t = 3\sin s$. The common point is $(3, 0)$ corresponding to $t = 1$, $s = 0$.

Alternatively: Eliminate t from the equations of both curves.

A 07

1. The curve is closed iff there is a common period of both $x(t)$ and $y(t)$. This is equivalent to the requirement that there exist integers n, m such that $\omega_1/\omega_2 = n/m$.
2. $\omega_1 = \omega_2$, $\varphi = 0$.
3. For T large enough the curve gets arbitrarily close to any fixed interior point of a rectangle Q defined by the values a, b, i.e. the graph is dense in the rectangle $Q = [-a, a] \times [-b, b]$ for $T \to \infty$. Indeed, let (x, y) belong to Q. We can find a value t_0 such that $a \sin \omega_1 t_0 = x$ and so $x = a \sin \omega_1 (t_0 + 2k\pi/\omega_1)$ for all integers k. Since ω_1/ω_2 is irrational the values $b \sin(\omega_2(t_0 + 2k\pi/\omega_1) + \varphi)$ are dense in $[-b, b]$.

A 08

$y - f(x_0) = f'(x_0)(x - x_0)$ and

$$x = tx'(t_0) + x(t_0),$$

$$y = ty'(t_0) + y(t_0),$$

$$z = tz'(t_0) + z(t_0), \quad t \in \mathbf{R}.$$

The half-tangent can be defined as the beam with the above equation and $t \geq 0$ or $t \leq 0$.

A 09

1. The points of the curve are only in the segments $t \in (2k\pi/n,\ (2k+1)\pi/n)$ and in each of these is increasing and decreasing symmetrically around its middle point $(4k+1)\pi/(2n)$. The curve resembles a symmetrically shaped flower with n equal leaves.
2. The curve in Cartesian coordinate is

$$x^2(1 - \epsilon^2) + y^2 + 2px\epsilon - p^2 = 0$$

and it represents an ellipse, parabola or hyperbola if $\epsilon < 1$, $\epsilon = 1$, $\epsilon > 1$, respectively.

A 10

The visible part of the ellipse consists of its points with
$x \in [-1.062740, 1.951629]$.

A 11

1. $N > \frac{\pi}{\arctan\sqrt{R^2-1}}$, where R is the distance from the center of the circle.
2. A lower estimate of N can be given when neglecting the invisibility of some parts of the sphere and henceforth by dividing the surface area of the sphere by the surface area of the part visible from a fixed point. This approach yields the estimate $N > (2R)/(R-1)$.

Another approach aims at calculation of R assuming N given. For $N = 2$ evidently $R = \infty$. With vertices of regular polyhedra as tangent points of 'visibility conuses' we obtain: For

$N = 4$ there is $R = 3$,

$N = 6$ there is $R = \sqrt{3}$,

$N = 8$ there is $R = \sqrt{3}$,

$N = 12$ there is $R = 1.25841$,

$N = 20$ there is $R = 1.17557$.

Here you may use data obtained from the package `Graphics`Polyhedra``.

A 12

In both cases 1 and 2 the half-tangents are parts of the x-axis, straight line of slope b/a or y-axis if $m > n$, $m = n$ and $m < n$, respectively. The parts of the arc corresponding to nonpositive and nonnegative values of t are both contained in the same half-plane or in opposite half-planes if $m - n$ is even, or $m - n$ is odd, respectively.
3. The slopes of the half-tangents are $-1/6$ and $1/4$ for $t > 0$ and $t < 0$, respectively.

A 13

1. The point moves along a segment from the point (b, d) to the point $(a + b, c + d)$ with decreasing velocity towards the endpoint.
2. The point moves along a segment with endpoints $(b - a, d - c)$, $(a + b, c + d)$, starting at (b, d) and passing each point except the endpoints twice.

A 14

The parabolas cross at $(1/a, 1/a)$ with the angle $\alpha = \arctan(3/4) = 0.643501$.

A 16

$$y = px, \quad p \geq 0.$$

A 18

All points of the curve are points on the cone $x^2 + y^2 = (z - 2)^2$. They have the form of a helix and α determines its ascent.

11

A 20

E.g. the arc $x(t) = ct^3 + (1-2c)t^2 + ct$, $y(t) = (1-2c/3)t^3 + (1+2c/3)t^2$, $0 \le t \le 1$ connects the given arcs for all $c > 0$. You may choose c according to your aesthetic feeling.

A 21

The coefficients of $x(t) = at^3 + bt^2 + ct + d$, $0 \le t \le 1$, and successively also the coefficients for $y(t)$ can be expressed in MATHEMATICA® as

```
koef[k_, q_, u_] := Module[{c = k (u[[2]] - u[[1]]),
   a = q (u[[4]] - u[[3]]) - 2 (u[[4]] - u[[1]]) +
     k (u[[2]] - u[[1]]),
   b = 3 (u[[4]] - u[[1]]) - 2 k (u[[2]] - u[[1]]) -
     q (u[[4]] - u[[3]])},
  {a, b, c, u[[1]]}]
```

where `u[[i]], i=1,2,3,4,` are successively the x coordinates, and the y coordinates of the four given points, respectively.
In Maple® analogously

```
F:=(k,q,u)->[c:=k (u(2)-u(1)),
a:=q (u(4)-u(3))-2 u(4)+2 u(1)+k (u(2)-u(1)),
b:=3 u(4)-3 u(1)-2 k (u(2)-u(1))-q (u(4)-u(3)),
d:=u(1)]
```

The constants $k, q > 0$ are arbitrary, for $k = q = 3$ this module yields the curves which are commonly called Bezier curves.

A 22

With $d = 5$, $u = 7$, $h \le 16$ a truncated conus has volume $V = 456.6$. We may try to design a bowl of suitable height h assuming its shape obtained by rotation of a Bezier curve with an appropriate choice of four basic points: After some experiments we may come to the following data and results (here $T = (t^3, t^2, t, 1)$, and koef is the procedure described above).

```
ParametricPlot[{T.koef[3, 3, {0, 4, 7.5, 16}],
T.koef[3, 3, {2.5, 2.35, 3.8, 3.5}]}, {t, 0, 1},
AspectRatio -> 0.5]
```

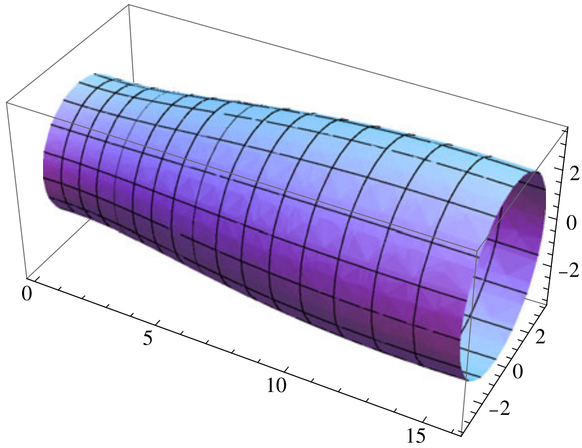
The volume V can be calculated as follows $V = \pi \int_0^1 y^2(t)x'(t)dt$. In our case we obtain $V = 499.86$, which is close to the desired value and the resulting height of the bowl equals 16. To find the calibration mark e.g. for $V/2$ the equation $V/2 = \pi \int_0^m y^2(t)x'(t)dt$ for the unknown value m has to be solved. We obtain $t \cong 0.698$, which corresponds to $x = 9.51$. After putting

```
x = T.koef[3, 3, {0, 4, 7.5, 16}]
y = T.koef[3, 3, {2.5, 2.35, 3.8, 3.5}]
```

we may obtain a view of the design by

```
ParametricPlot3D[{x,y Cos[u],y Sin[u]},{t,0,1},{u,0,2\[Pi]}]
```

thus



A 23

A possible solution can be obtained in MATHEMATICA® by

```
ParametricPlot3D[{T.koef[3, 3, {0, -1, 3, 2}],
T.koef[3, 3, {0, 1, 2, 3}],
T.koef[3, 3, {0, 1, 2, 3}]}, {t, 0, 1}]
```

as $(x, y, z) = (-3t + 15t^2 - 10t^3, 3t, 3t)$.

A 24

Choose a point of the interior of the domain as the origin, transform the coordinates into the polar system (r, φ) and express the area as

$$P = \frac{1}{2}\sum_{i=1}^{n-1} r_i r_{i+1} \sin\gamma_i, \quad \text{where } \gamma_i = \varphi_{i+1} - \varphi_i.$$

Alternatively, use the results of A 35.

A 25

The slope of the tangent at the point (x_0, y_0) follows from

$$y' = \frac{y_0 - x_0^2}{y_0^2 - x_0}.$$

For $x = 1/2$ we obtain for y three real solutions -1.26446, 0.0837246, 1.18073, and the slopes of the tangents are -1.37821, 0.337279, 1.04093, respectively. One of the tangents has the equation $y + 1.26446 = -1.37821(x - 0.5)$.

11

A 26

$$y = x\cos t - t\cos t + \sin t.$$

A 27

(a) Parabola $y = x^2$, tangents are at the points $x = \lambda$;
(b) hyperbola $y^2 - x^2 = 1$, tangents are at the points $x = \tan \nu$;
(c) upper part of the circle $x^2 + y^2 = 1$, tangents are at the points $x = \cos t$, $y = \sin t$.

A 28

For the set of straight lines $(1+\lambda^2)y + 2\lambda x - 1 = 0$ we obtain $y^2 - x^2 - y = 0$.

A 29

The concept of the length of a curve does not satisfy the 'natural' requirement of continuity demanding that 'close' curves must have 'close' lengths.

A 30

The function g is $g(t) = \int_0^t \sqrt{\dot{x}^2(u) + \dot{y}^2(u)}du$. In many cases the function g cannot be expressed in terms of elementary functions.

A 31

$x = \log(s + \sqrt{1+s^2})$, $y = \sqrt{1+s^2}$ and

$$x = 2\arctan\frac{e^s - 1}{e^s + 1}, \qquad y = -\log\cosh s.$$

The result can be checked easily: it must be $\dot{x}^2 + \dot{y}^2 = 1$.

A 32

The river must be replaced by its mathematical model—an arc, which has a length by definition. Often the so called middle stream is used. It can be defined as the set of points such that each point has the same distance from both banks, however this definition is not commonly accepted.

A 33

If the arc starts at the point (a, a^2), then the midpoint x satisfies the equation

$$2a\sqrt{1+4a^2} + d = 2x\sqrt{1+4x^2} + \log\frac{2x + \sqrt{1+4x^2}}{2a + \sqrt{1+4a^2}}.$$

Newton's method of solution of the resulting equation is appropriate. For $a = 0$, $d = 1$ we obtain $x = 0.446334$.

A 34

For arcs $(x(t), y(t))$ with an arc length parametrization, i.e. such that $\dot{x}^2 + \dot{y}^2 = 1$ for all $t \in I$ the endpoints correspond to the value $t = kd/n$, $k = 1, 2, \ldots, n-1$ where d is the length of the interval I.

Approximate values can be found by plotting the graph of the primitive function of $\sqrt{\dot{x}^2 + \dot{y}^2}$ and its inverse. The values of this inverse at successive equidistant points are the required values of t.

A 35

For a closed arc $(x(t), y(t))$, $t \in I$ the area A can alternatively be expressed as

$$A = \int_I x(t)y'(t)dt = -\int_I x'(t)y(t)dt = \frac{1}{2}\int_I (x(t)y'(t) - x'(t)y(t))dt.$$

A 37

The curve is the graph of a function $y = y(x)$, which satisfies the differential equation

$$y' = \frac{-y}{\sqrt{k^2 - y^2}}, \qquad y(0) = a,$$

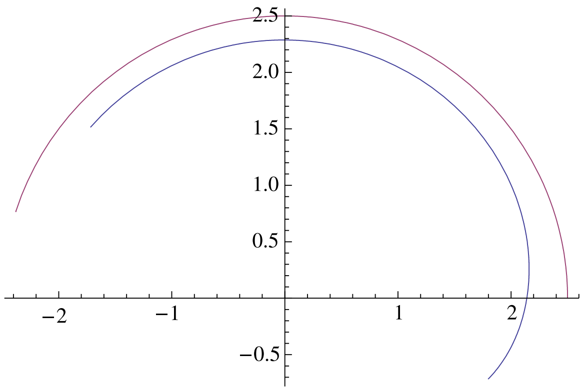
where k is the length of the rope. Integration yields

$$x = -\sqrt{k^2 - y^2} + k \log \frac{2(k + \sqrt{k^2 - y^2})}{k^2 y} + C,$$

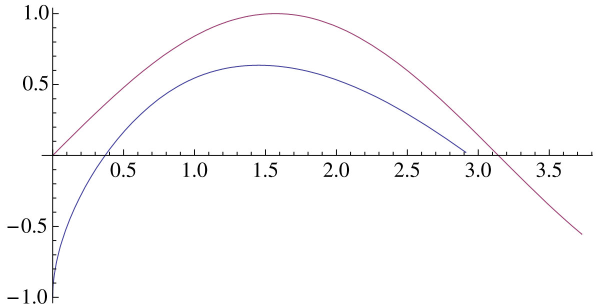
where $C = \sqrt{k^2 - a^2} - k \log \frac{2(k+\sqrt{k^2-a^2})}{k^2 a}$. Evidently, $k \geq a$.

A 39

(a) With $\zeta = 2.5\cos t$, $\eta = 2.5\sin t$, $0 \leq t \leq \pi$ and $x(0) = 1.8$ we obtain $y(0) = -0.714143$. Numerical solution gives the following result



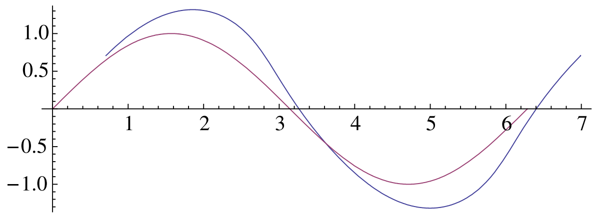
(b) With $\xi = t$, $\eta = \sin t$, $0 \le t \le 3\pi$ we obtain



(c) With the rear wheel on the track $y = \sin x$ and with the starting point of the front wheel at $(1/\sqrt{2}, 1/\sqrt{2})$ we obtain

$$(\xi, \eta) = \left(x + \frac{1}{\sqrt{1+\cos^2 x}}, \frac{\cos x}{\sqrt{1+\cos^2 x}} + \sin x \right),$$

i.e.

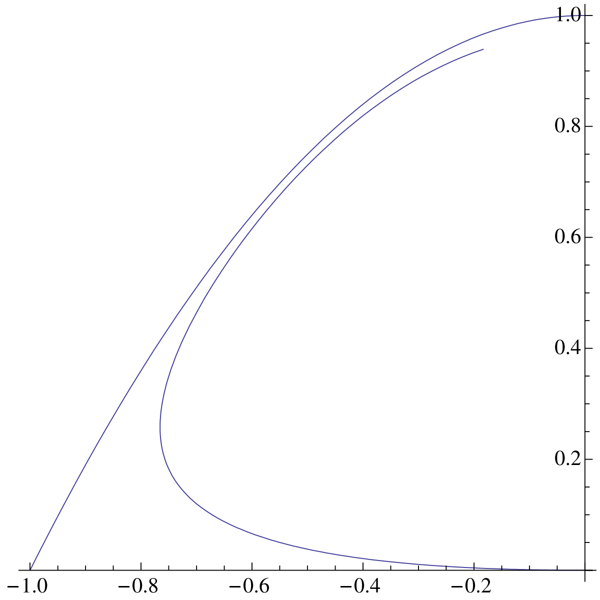


A 42

For a parabola $\xi(t) = t$, $\eta(t) = 1 - t^2$ and $v(t) = 5\sqrt{1+4t^2}$ the calculation has to use numerical solution of the differential equation.
In MATHEMATICA®

```
 NDSolve[{x'[t] + 5 Sqrt[1+4 t^2] x[t] == 5 t Sqrt[1+4 t^2],
y'[t]+5 Sqrt[1+4 t^2] y[t] == 5 (1-t^2) Sqrt[1+4 t^2],
x[-1] == 0,y[-1] == 0},{x,y},{t,-1,0}]}
```

In Maple® typing this differential equation into the `ODE Analyzer Assistant` gives the pursuit curve and the parabola as follows



Center of Mass and Moments (C)

C 01

With $A_i = (x_{i1}, x_{i2}, \dots x_{iN})$, $i = 1, 2, \dots, n$ the coordinates in $\mathbf{R}^N$ are

$$x_{ck} = \frac{1}{\sum q_i} \sum_{i=1}^{n} q_i x_{ik}, \quad k = 1, 2, \dots, N$$

and $z_c = \frac{1}{n} \sum_{i=1}^{n} q_i z_i$ for points z_i in the complex plane.

C 03

(a) $T^+ - T = \frac{1}{n+1}(r - T)$, where r is the radius-vector of the $(n+1)$st point.
(b) $T^+ - T = \frac{q_{n+1}}{\sum_{k=1}^{n+1} q_k}(r - T)$, where q_k is the mass of the k-th point.

C 04

Make the straight line l one of the coordinate axes in the direction of a basis vector e. Then the contribution of each pair of symmetric points is a multiple of e.

C 05

$$r_{A\cup B} = \frac{q_A r_A}{q_A + q_B} + \frac{q_B r_B}{q_A + q_B}.$$

For the possibly infinite set A_n, $n \geq 1$, $r = \sum \frac{q_{A_i}}{\sum q_{A_i}} r_{A_i}$ provided all r_{A_i} exist and $q_{A_i} < \infty$.

C 06

For $p(z) = \sum_{k=0}^{n} a_k z^k$ the sum of roots equals $-\frac{a_{n-1}}{a_n}$, hence the condition reads $a_{n-1} = 0$.

The transformation is $z \to z - \frac{a_{n-1}}{na_n}$.

C 10

Let d be the diameter of the circle passing through the given point A. A chord perpendicular to d and passing through A meets the circle at the desired positions. This solution is unique.

C 11

Let the circle of radius r have its center at the origin and let the prescribed point $P = (p, 0)$, $p > 0$. Then the three points in the complex plane are

$$z_1 = r, \qquad z_{2,3} = re^{\pm \mathbf{i}\varphi} \quad \text{with } \cos\varphi = \frac{3p - r}{2r}.$$

C 12

To place 6 points so that their center of mass is at the point $z = 0.25$ we have to find three values such that their arithmetic mean equals 0.25, e.g. $v = \{0.5, 0.2, 0.05\}$. These are the real parts of the six points $z_k = \exp \mathbf{i}\, t_k$ with $t_k = \pm \arccos v_k$.

To place 7 points so that their center of mass is at the point $z = 0.25$ we may place one point at $z = -1$ and proceed according to C 03. We obtain again $z_k = \exp \mathbf{i}\, t_k$ with $t_k = \pm \arccos v_k$, where e.g. $v = \{-1, 0.5, 0.2, 0.675\}$.

C 13

Let the prescribed point be at the origin, let r be the radius of the circle and let $(d, 0)$, $r > d > 0$ be the coordinates of the center.

If $p = q$ then the points are $(0, \pm\sqrt{r^2 - d^2})$.

If $p \neq q$, the straight line $y = k(x - d)$, with

$$k^2 = \frac{1}{r^2 - d^2}\left(\left(\frac{p+q}{p-q}\right)^2 d^2 - r^2\right)$$

forms the chord. The problem has no solution iff $d < r\left|\frac{p-q}{p+q}\right|$.

A direct solution of the equation

$$pr\exp(\mathbf{i}\alpha) + qr\exp(\mathbf{i}\beta) = d(p+q)$$

together with its conjugate for the unknowns α, β is also possible.

For $p = 2,\ q = 3,\ r = 2,\ d = 1,$ the coordinates of the pair of points are $(\frac{1}{4}, \frac{-3\sqrt{7}}{4})$, $(\frac{3}{2}, \frac{3\sqrt{7}}{2})$ or a pair symmetric with respect to the x-axis.

C 14

Place a mass $m = q/20$ at the point on the circle opposite to the given center of mass.

C 16

Let $f(\phi) = \frac{L_1}{L_2}$, where L_1, L_2 are the lengths of the opposite segments of the chord passing through the origin, and ϕ denotes the angle between the chord and the x-axis. If $f(\phi_0) < 1$ for some ϕ_0 then $f(\phi_0 + \pi) < 1$. Hence there is a ϕ such that $f(\phi) = 1$.

C 17

For the proof see PC 12.

C 18

The solution is not unique. One of the solutions is (0, 1), (1.5, 0.5), (−1.7247, −0.0917), (0.7247, −0.9082), (−1.5, 0.5), (1.5, −0.5).

C 21

Any $n + 1$ mutually different moments determine the polynomial. With a given moment m_k and a polynomial $P_n(x)$ with coefficients a_i the equation $m_k = \int_0^1 x^k P_n(x)dx$ is an equation for the unknown coefficients.

C 22

For $f(x) = \sin\pi x,\ -1 \le x \le 1$ and $n = 6$ we obtain the moments of f as

$$\{0.636619772367581, 0, 0.249601359169187, 0, 0.1308216693724141, 0\}$$

and the polynomial $3.103460428x - 4.814388319x^3 + 1.726905184x^5$ with a warning that the results may contain significant numerical errors.

On the interval $[-0.99, 0.99]$ the values of this polynomial differ from the values of $\sin\pi x$ by at most $7 \cdot (10^{-3})$.

C 23

For $f(x) = \sin\pi x,\ -1 \le x \le 1$ and $n = 6$ we obtain identical results with C 22 without the warning of possible errors.

11

C 25

Denoting by M_a and M the vector of moments (in ascending order) with centers in a and 0, respectively, we obtain $M_a = QM$, where Q is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ -a & 1 & 0 & 0 & 0 & \dots \\ a^2 & -2a & 1 & 0 & 0 & \dots \\ -a^3 & 3a^2 & -3a & 1 & 0 & \dots \\ a^4 & -4a^3 & 6a^2 & -4a & 1 & \dots \\ \dots & \dots & \dots & \dots & & \end{pmatrix}.$$

Inverting this matrix of fixed order n we express M in terms of M_a. For Q^{-1} we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ a & 1 & 0 & 0 & 0 & \dots \\ a^2 & 2a & 1 & 0 & 0 & \dots \\ a^3 & 3a^2 & 3a & 1 & 0 & \dots \\ a^4 & 4a^3 & 6a^2 & 4a & 1 & \dots \\ \dots & \dots & \dots & \dots & & \end{pmatrix}.$$

C 27

The first moment of the data $\mu = 1.957835371518626$, the second central moment $\sigma = 1.2844761958156585$.

C 31

With $z = \exp \mathbf{i} t$, $-\alpha/2 \le t \le \alpha/2$ the coordinates (ξ, η) of the center of mass are $(\frac{2\sin\alpha/2}{\alpha}, 0)$.

The arc transformed into a new coordinate system (with the endpoint of the arc at the origin and freely suspended) has the equation

$$w = (\exp(\mathbf{i}\phi) - \exp(\mathbf{i}\alpha/2))\exp(-\mathbf{i}t)$$

with $\phi \in (-\alpha/2, \alpha/2)$ and $\tan t = \frac{2}{\alpha} - \cot(\alpha/2)$. The requested endpoint w in the complex plane is $w = \frac{-2\mathbf{i}\sin\alpha}{2}\exp(-\mathbf{i}\phi)$ with $\tan\phi = \frac{2}{\alpha} - \cot\alpha$ provided $0 < \alpha < \pi$.

E.g. for $\alpha = \pi/3$, $q = 1$ we obtain $T = (0.6118264129568, -1.6203914466598)$ and for $\alpha = \pi/2$ there is $T = (-0.3202767645490, -1.8197901957267)$.

C 32

Let the cylinder be given as $x^2 + y^2 + rx = 0$, $0 \le z \le h$. The water level is characterized by the plane of equation $z = kx + hk = \tan\alpha$. We obtain for the center of mass the point

$$(x_c, y_c, z_c) = \left(-\frac{r}{2} + \frac{r^2 k}{8(2h - kr)}, 0, \frac{h}{2} + \frac{rk(8h - 3kr)}{16(2h - kr)}\right).$$

The critical angle α is determined by the equation

$$x_c \cos\alpha + z_c \sin\alpha = 0$$

which leads to a cubic equation for the unknown k.

C 34

$$\frac{\sum_{i=1}^{n} c_i q_i}{\sum q_i} + \frac{\int_0^1 x\rho(x)dx}{\int_0^1 \rho(x)dx}.$$

C 42

$$B = ((3-\sqrt{3})/6, (3-\sqrt{3})/6) = (0.2113248654, 0.2113248654).$$

C 44

The answer is 'no'. The right function to minimize is given as

$$h(\xi,\eta,\zeta) = \int_0^1 [(x(t)-\xi)^2 + (y(t)-\eta)^2 + (z(t)-\zeta)^2]\sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}dt.$$

Miscellaneous (X)

X 01

The polynomial must be of even degree otherwise one of the limits at $x \to \pm\infty$ is negative. Such polynomials must be of the form of a product of quadratic polynomials with $b^2 - 4ac \le 0$ and $a > 0$.

X 02

$y(x) = 1/3(1/x^2 - 2x)$.

X 03

$y(x) = \frac{2\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x-1}-4\sqrt{x(x+1)}}$ for all $x > 1$.

X 04

1. y is only the function constant on I.
2. $y = \arccos(1/\sqrt{5}) + 2k\pi$ for all $x \in \mathbf{R}$ and any integer k. No other solutions exist.
3. The equation has no solution.

11

X 05

Put $\omega_k(x) = \prod_{i=1,i\neq k}^{n}(x - a_i)$ for $k = 1, 2, \ldots, n-1$. For the numerator $c(x)$ of $f(x)$ we obtain $c(x) = \sum_{i=1}^{n} A_i\omega_i(x)$. Calculate $\operatorname{sign} c(a_k)$. We have $\operatorname{sign} c(a_k) = (-1)^r$, where r is the number of a_i less than a_k. Evidently $\operatorname{sign} c(a_k)\operatorname{sign} c(a_{k+1}) = -1$ and therefore between any two consecutive values of a_k there is at least one real zero of $c(x)$. Since the degree of $c(x)$ equals $n-1$, the number of such zeros cannot be greater than 1, which has to be proved.

X 06

Put $\omega_k(x) = \prod_{i=1,i\neq k}^{n}(x^2 + a_i^2)$ for $k = 1, 2, \ldots, n-1$. Then the numerator $c(k)$ of $f(x)$ is of the form $c(x) = \sum_{i=1}^{n} A_i x\omega_i(x)$. The values of $\operatorname{sign}(-\mathbf{i}\, c(\mathbf{i}\, a_k)) = (-1)^r$, where r is the number of a_i less then a_k. Proceed as in X 05.

X 07

We may assume that between a and b the polynomial P_n has no other zeros. Then $P_{n+1}(a)P_n'(a) < 0$, $P_{n+1}(b)P_n'(b) < 0$ and $P_n'(a)P_n'(b) < 0$. Hence $P_{n+1}(a)P_{n+1}(b) < 0$ and so there exists $c \in (a, b)$ with $P_{n+1}(c) = 0$.

X 08

Consider the fact that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \geq \left(\prod_{k=1}^{n} f\left(\frac{k}{n}\right)\right)^{\frac{1}{n}} \geq \frac{n}{\sum_{k=1}^{n}\frac{1}{f(\frac{k}{n})}}.$$

Applying Jensen's inequality with $\Phi(x) = \log x$ we obtain the latter inequality. With $\Phi(x) = \log x$ applied to $1/f(x)$ we obtain the former one.

X 09

With $c = \frac{a_1}{d}$ we have $\log\frac{G_n}{2A_n} = \frac{1}{2n}\sum_{k=1}^{n-1}\log(\frac{c+k}{2c+n-1})$. The partition of [0, 1] given by the points $\frac{c+k}{2c+n-1}$, $k = 0, 1, \ldots, n-1$, with $c > 0$ arbitrary, can be used for calculation of the integral $\int_0^1 \log x\, dx = -1$.

X 12

The ellipse is $\mu^2x^2 + 3\lambda^2y^2 = \lambda^2\mu^2$ and it satisfies the requirements, e.g. the foci are at the square roots of $\lambda^2 - \mu^2/3$.

X 13

$$q^2x^2 + (3 + p^2)y^2 - 2pqxy - 2qy = 0.$$

X 14

The foci are at the points with coordinates

$$\left(\frac{p}{3} + \frac{\sqrt[4]{S}(3 + p^2 - q^2 + \sqrt{S})}{6pq\sqrt{1 + \frac{(3+p^2-q^2+\sqrt{S})^2}{4p^2q^2}}}, \frac{q}{3} + \frac{\sqrt[4]{S}}{3\sqrt{1 + \frac{(3+p^2-q^2+\sqrt{S})^2}{4p^2q^2}}}\right)$$

and

$$\left(\frac{p}{3} - \frac{\sqrt[4]{S}(3 + p^2 - q^2 + \sqrt{S})}{6pq\sqrt{1 + \frac{(3+p^2-q^2+\sqrt{S})^2}{4p^2q^2}}}, \frac{q}{3} - \frac{\sqrt[4]{S}}{3\sqrt{1 + \frac{(3+p^2-q^2+\sqrt{S})^2}{4p^2q^2}}}\right),$$

where $S = p^4 + (-3 + q^2)^2 + 2p^2(3 + q^2)$. The polynomial $M(z)$ with zeros at these points can be written as $(-1 - 2pz - 2\mathbf{i}qz + 3z^2)$, which is a derivative of $(z - 1)(z + 1)(z - p - \mathbf{i}q)$.

X 15

Denoting by I_n the interval with endpoints $\alpha_n = f^{-1}(a_n)$ and $\beta_n = f^{-1}(b_n)$, we obtain that $I_{n+1} \subset I_n$ and length $I_{n+1} = |s - t|$ length I_n. Hence the limit of both α_n and β_n exists and equals

$$\bigcap_{n=1}^{\infty} I_n = \frac{s\alpha_0 + (1 - t)\beta_0}{s + 1 - t}.$$

X 16

$\alpha_{n+1} = t\alpha_n + (1 - t)\beta_n$, $h(\beta_n) = sh(\alpha_n) + (1 - s)h(\beta_n)$.

Denoting by I_n the interval with endpoints $\alpha_n = f^{-1}(a_n)$ and $\beta_n = f^{-1}(b_n)$, we obtain that $I_{n+1} \subset I_n$ and length $I_{n+1} \le \max\{t, 1 - t\}$ length I_n.

X 17

For $x = 1/\lambda$ we obtain for successive values of $N \ge 2$ the following results: $x = 0.618034$, $x = 0.543689$, $x = 0.51879$, $x = 0.50866$, $x = 0.5014138$, $x = 0.502017$, $x = 0.500994$.

Inequalities 5

Quantitative descriptions of physical, technical and economical phenomena include quantities which are inherently positive. Their mathematical models are adequate only if this fact is reflected by the mathematical structure involved. Positivity, i.e. a special inequality, is therefore an important issue in analysis and applications.

On the other hand, the square of any real number is nonnegative. This fact alone is both a source and a method of proof for many important inequalities, which are widely used in analysis and its applications.

In this chapter you will find some inequalities which are often used in mathematical reasoning and their knowledge is indispensable. Also methods of deriving new inequalities and estimates are included. Last but not least, applications of inequalities in solving mathematical problems arising outside mathematics are included.

The solution of problems marked [M] and many others can be made easier by using software packages of S. Wolfram's MATHEMATICA®, although only very few of them can be solved directly. The user is strongly advised to learn the syntax and semantics of some of the commands. In this chapter the following commands in MATHEMATICA® (and related ones) may help:
`Plot, ListPlot, Plot3D, ContourPlot, LinearProgramming, FindRoot`

Similarly in Maple®:
`plot, plot3d, plots[contourplot], Optimization[LPSolve], solve, Student[NumericalAnalysis][Roots]`

In most of the examples numerical experiments may give a starting point for reasoning or verify initial conjectures.

Explanation of mathematical terms and concepts can also be found on the Internet, e.g. at www.mathworld.wolfram.com or at http://en.wikipedia.org/wiki/Portal:Mathematics or at http://eom.springer.de.

J. Gregor, J. Tišer, *Discovering Mathematics*,
DOI 10.1007/978-0-85729-064-9_6,

5

Suggestions

- Indispensable even in elementary calculus are the results of E 01, E 02, E 04, E 05 and useful methods are presented in E 10, E 13, E 36. Continue further along the downward arrows.
- Those who are research inclined will find interesting stimuli in E 16, E 19, E 23, E 24, E 28, E 34.
- Teachers could use E 18, E 25, E 29, E 30, E 40 to enhance understanding and for broader context.
- All readers are strongly encouraged to modify, generalize or simplify the formulated problems, to find alternative formulations, formulate and solve their own examples and compare the context of these problems to the given ones.

Problems

E 01 • $\downarrow$ **I 63** $\downarrow$ **E 03** $\downarrow$ **E 04** $\downarrow$ **E 19** $\downarrow$ **E 31** $\downarrow$ **E 32** $\downarrow$ **E 33** $\downarrow$ **E 34** $\downarrow$ **E 35**

Show that the arithmetic mean of any two positive numbers is never smaller than their geometric mean, i.e. that

$$a + b \geq 2\sqrt{ab}.$$

E 02 • $\uparrow$ **E 01**

Prove the following inequality for any two real numbers a, b and find its geometric interpretation:

$$\sqrt{a^2 + b^2} \leq |a| + |b|.$$

E 03 • $\uparrow$ **E 01** $\downarrow$ **E 32**

Prove that the harmonic mean of any two positive numbers is never greater then their geometric mean, i.e. that

$$\frac{2ab}{a+b} \leq \sqrt{ab}.$$

E 04 •

Find the necessary and sufficient conditions concerning the real numbers a, b which would ensure equality in the inequalities of E 01, E 02, E 03.

E 05 •• ↑ **E 01** ↑ **E 02**

Let a, b be complex numbers. Find a proof of the triangular inequality

$$||a| - |b|| \leq |a + b| \leq |a| + |b|.$$

Find the necessary and sufficient conditions for a, b such that

$$|a + b| = |a| + |b|.$$

Prove also the following generalization:

$$\left|\sum_{i=0}^{n} a_i\right| \leq \sum_{i=0}^{n} |a_i|.$$

Hint: For the right-hand side inequality consider $|a + b|^2$, use the fact that $|a|^2 = a\bar{a}$ and recall that $\operatorname{Re} z \leq |z|$ for any complex z, with ($\bar{a} = \text{Conjugate}[a]$). For the left-hand side inequality put $b - a$ instead of b.

This is probably the most important and most frequently used inequality. The number of its generalizations as well as its use in proofs, definitions and applications can hardly be overestimated.

E 06 •

Find conditions for two positive numbers a, b ensuring that $a^b < b^a$.

Hint: Try to reduce the problem to investigation of the function

$$f(x) = \frac{\log x}{x}.$$

5

E 07 ●● ↑ E 06

It can be said that the exponential function of a single real variable grows faster than any power of the variable. Give a precise formulation of this statement and prove it.

E 08 ●●

Let $a_1, a_2, \ldots, a_n, b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ be two sequences of real numbers and let

$$s_k = a_1 + a_2 + \cdots + a_k, \qquad m = \min s_k, \qquad M = \max s_k.$$

Then

$$mb_1 \leq a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq Mb_1$$

(Abel's inequality). Find a proof.

Hint: Express the sum $\sum_{k=1}^{n} a_k b_k$ in terms of b_k, s_k.

E 09 ● [M]

Characterize the set A composed of points $(x, y) \in \mathbf{R}^2$ satisfying the inequalities:

1. $|x| + |y| \leq 1$,
2. $\max(|x|, |y|) \leq 1$.

Characterize the set A composed of points $(x, y, z) \in \mathbf{R}^3$ satisfying the inequalities:

3. $|x| + |y| + |z| \leq 1$,
4. $\max(|x|, |y|, |z|) \leq 1$.

Hint: Note the symmetry $(x, y) \in A \Rightarrow (\pm x, \pm y) \in A$ and similarly for $\mathbf{R}^3$.

E 10 •• ↓ **E 11**

Prove the inequality

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

Hint: Consider the nonnegative quantity $\sum_{i=1}^{n}(x_i u + y_i)^2$ in a variable u and look at the discriminant.

E 11 • ↑ **E 10**

Let f, g be square integrable functions on (a, b). Prove that

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx.$$

Hint: Consider $\int_a^b (f(x)u + g(x))^2 dx$ for a variable u.

Inequalities in E 10, E 11 are special cases of the important inequality named after Cauchy, Schwarz and Bunjakovskij, which is widely used and cited under one or two of these names.

E 12 ••

Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n)$ be n-tuples of real numbers such that both are increasing or both are decreasing. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \leq \left(\frac{1}{n}\sum_{k=1}^{n} a_k b_k\right)$$

(Chebyshev's inequality).

Hint: Denote $\sum a = \sum_{k=1}^{n} a_k$ and consider $n\sum ab - \sum a \sum b$ as a double sum which can be expressed in terms of $(a_k - a_j)(b_k - b_j) \geq 0$.

E 13 • **[M]** ↓ **E 15** ↓ **E 16** ↓ **E 17** ↓ **E 18**

For $x \in [0, \pi/2]$ prove that $\frac{2}{\pi}x \le \sin x \le x$ (Jordan's inequality).

Hint: Analyze the behavior of $f(x) = x - \sin x$ and $g(x) = \sin x - \frac{2}{\pi}x$.

E 14 •

Suppose that two functions f, g are defined on the same interval T and $f(x) \le g(x)$ for all $x \in T$. Prove or disprove the following statements:

(a) $f'(x) \le g'(x)$,
(b) $\int_y^x f(x) \le \int_y^x g(x)$, $y < x$.

E 15 • **[M]** ↑ **E 13**

Show that

$$x - \frac{x^3}{6} \le \sin x \quad \text{and} \quad 1 - \frac{x^2}{2} \le \cos x \quad \text{for } 0 \le x \le \pi/2.$$

Hint: Proceed with a direct proof similar to the one in E 13.

E 16 •• **[M]** ↑ **E 13**

For $x \ge 0$ and $p \ge 1$ prove that $(1 + x)^p \ge 1 + px$. Is it valid in other cases?

Hint: The proof is easy for integer values of p. Otherwise use the method of E 13.

E 17 • **[M]** ↑ **E 13**

Find all values of x for which the inequality $\frac{1}{1+x^2} \ge e^{-x^2}$ holds true.

Hint: It may help to plot the two functions.

E 18 •• [M] ↑ E 13 ↓ E 26

Find a proof of the following inequalities provided that $x > 0$:

$$x^a - ax + a - 1 \leq 0 \quad \text{if } 0 < a < 1,$$
$$x^a - ax + a - 1 \geq 0 \quad \text{for other values of } a.$$

Use this result to derive the inequality

$$x^a y^{1-a} \leq ax + (1-a)y, \quad 0 < a < 1,$$

and show that the inequality in E 01 is its special case.

Hint: Consider the derivative of the left-hand side of the inequalities. To prove the third one use x/y instead of x in the first inequality.

E 19 ••

Let a nonnegative function f have a continuous derivative on the interval $[a, b]$ and let $f(a) = f(b) = 0$. Then there must exist a number $\xi \in [a, b]$ such that

$$f'(\xi) \geq \frac{4}{(b-a)^2} \int_a^b f(x)dx.$$

Find a proof.

Hint: There exists a ξ such that $|f'(\xi)| = \max_{x\in[a,b]} |f'(\xi)| = M$. Then $f(x) \leq M(x-a)$ for all $a \leq x \leq \frac{a+b}{2}$ and $f(x) \leq M(b-x)$ for all $\frac{a+b}{2} \leq x \leq b$ and the result follows.
Plot a graph representing these inequalities.

E 20 •

Assume that in E 19 the function f is the velocity of motion and find a physical interpretation of the inequality therein.

5

E 21 • ↓ **E 22**

Give a geometric interpretation of a convex function in terms of its graph and chords passing through the two points $(x_1, f(x_1))$, $(x_2, f(x_2))$.

Hint: In the definition of a convex function consider first the geometric meaning of equality.

E 22 • ↑ **E 21** ↓ **E 25**

Suppose that f and g are convex functions on a closed interval $[a, b]$. For each of the functions h_i described below determine whether it is convex:

$$h_1 = f + g, \qquad h_2 = fg, \qquad h_3 = \int_a^x f(t)dt, \qquad h_4 = f(g),$$

provided that the composition of f and g is defined. Find examples and counterexamples.

E 23 •• ↑ **E 14**

Let f and g be positive functions with continuous and nonnegative derivatives on $[0, b]$ and let $f(0) = 0$. Then for $0 < a \leq b$

$$f(a)g(b) \leq \int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx.$$

Prove this statement.

Hint: Integrate by parts the first integral on the right-hand side.

E 24 • **M 24**

Let f be a continuous and strictly increasing function on $[0, c], c > 0$. If $f(0) = 0, a \in [0, c], b \in [0, f(c)]$ then

$$\int_0^a f(x)dx + \int_0^b f_{-1}(x)dx \geq ab,$$

where f_{-1} is the inverse function of f.
Verify this statement (Young's inequality) by its geometric interpretation.

E 25 •• **↑ E 18 ↑ E 21 ↓ E 26 ↓ E 33 ↓ E 34 ↓ E 35**

Suppose that the function f is convex on a closed interval $[a, b]$ and that $a \leq x_i \leq b$, $i = 1, 2, \ldots, n$, $\alpha_i > 0$, $\sum \alpha_i = 1$. Then

$$\sum_{i=1}^{n} \alpha_i f(x_i) \geq f\left(\sum_{i=1}^{n} \alpha_i x_i\right).$$

Find a proof. Formulate a similar result for concave functions.

Hint: Use induction over n. Assuming the validity for n, write

$$\sum_{i=1}^{n+1} \alpha_i x_i = \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + \hat{\alpha}\hat{x},$$

where

$$\hat{\alpha} = \alpha_n + \alpha_{n+1} \quad \text{and} \quad \hat{x} = \frac{\alpha_n x_n}{\hat{\alpha}} + \frac{\alpha_{n+1} x_{n+1}}{\hat{\alpha}}.$$

E 26 ●● ↑ E 01 ↑ E 25

Prove that the inequality

$$\log\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{1}{n}\sum_{i=1}^{n} \log x_i$$

for $x_i > 0$ follows from concavity of log.
Deduce from the inequality above that the arithmetic mean of n positive numbers is never smaller than their geometric mean, i.e. that

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \quad \text{for } x_i > 0.$$

Hint: The first part can be accomplished by induction.

E 27 ●● ↑ E 01 ↑ E 18 ↑ E 26

The concept of convex functions enables to find and prove a number of important inequalities, e.g. as in the following statements:
1. For nonnegative values of x, y, for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ there is

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q} \quad \text{or} \quad xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

2. Prove that

$$\log\left(\frac{\exp(-x)}{p} + \frac{\exp(-y)}{q}\right) + \frac{x}{p} + \frac{y}{q} \geq 0$$

for all real x, y and deduce a similar inequality with n summands.

Hint: 1. Put $x = \exp(\alpha)$, $y = \exp(\beta)$ and use the convexity of the exponential function.
2. In a more general inequality

$$\log\sum_{i=1}^{n} \frac{\exp(-x_i)}{a_i} + \sum_{i=1}^{n} \frac{x_i}{a_i} \geq 0,$$

with $\sum a_i = 1, a_i > 0$ we may recognize the inequality of E 26 (with a special choice of a_i).

E 28 • ↓ E 29 ↓ E 30

Characterize the set A composed of points (x, y) satisfying the inequalities $2x + 3y \le 3$, $x - y \le 1$, $-2x + y \le 4$.
Find the maximum value of $f(x, y) = x^2 + y^2$ and $g(x, y) = 3x + 4y$ across the set A. Can you change one of the coefficients in the inequalities so that the problem would have no solution?

E 29 •• ↓ PE 29 ↑ E 28

Suppose that two types of products P_1, P_2 can be produced using four types of raw material S_i, $i = 1, 2, 3, 4$, where up to b_i amounts of each can be used. The amount of raw material to be used equals a_{ik} for the i-th product and k-th material. The revenue of a unit of each of P_i equals c_i. We want to organize the production so as to maximize the revenue.
Denoting x_1, x_2 the quantities of production for P_1 and P_2, respectively, formulate the conditions under which the maximum of the function $F = c_1x_1 + c_2x_2$ for unknown x_1 and x_2 is to be found.
Find the solution of the problem with the following data:

$$c_1 = 7, \qquad c_2 = 5, \qquad (b_1, b_2, b_3, b_4) = (19, 13, 15, 18),$$
$$(a_{11}, a_{12}, a_{13}, a_{14}) = (2, 2, 0, 3),$$
$$(a_{21}, a_{22}, a_{23}, a_{24}) = (3, 1, 3, 0).$$

E 30 •• ↓ PE 30 ↑ E 28

Assume that we have to distribute from two places A_1, A_2 the amounts a_1 and a_2 tons of some material to three destinations B_i, which demand b_i tons each. The transportation costs per unit from A_i to B_j are known to be c_{ij}. The overall transportation costs are therefore given by

$$F = \sum_{i,j} c_{ij}x_{ij},$$

where x_{ij} denotes the amount to be transported from A_i to B_j. Assume that $a_1 + a_2 = b_1 + b_2 + b_3$. Find the values x_{ij} so as to minimize the function F.
Put $(b_1, b_2, b_3) = (10, 30, 10)$, $(a_1, a_2) = (20, 30)$ and $F = 4x_{11} + 9x_{12} + 3x_{13} + 4x_{21} + 8x_{22} + x_{23}$.

Problems E 28, E 29, E 30 are extremely simplified examples of so called linear programming problems. Using the notation

$$a \geq 0 \Longleftrightarrow a_i \geq 0 \quad \text{for all } i = 1, 2, \ldots, n$$

for n-dimensional vectors, it can be formulated as follows:
Let A be a matrix with n rows and m columns, $m > n$. Among positive solutions of the system of linear inequalities

$$Ax - b \geq 0$$

find the one which gives minimum value to the linear form

$$F = \sum_{i=1}^{m} a_i x_i$$

with given coefficients a_i. Here $x = (x_1, x_2, \ldots, x_m)$, and similarly for b.
These types of problems often occur in planning, e.g. in resource allocation, transport planning where often the dimensions of vectors and matrices are several tens or hundreds. Special methods have been developed for these purposes; the so-called simplex method is the most important.

E 31 ●● ↑ **E 01** ↑ **E 25**

If the product of n positive numbers equals 1 then their sum cannot be smaller than n.
Prove this statement.

Hint: Apply E 25 for $f(x) = \exp(x)$.

E 32 ● ↑ **E 01** ↑ **E 18** ↑ **E 26** ↑ **E 27**

For $\frac{1}{p} + \frac{1}{q} = 1$ and $a > 0, b > 0$ prove that

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \geq \frac{a}{p} + \frac{b}{q} \quad \text{for } p < 1.$$

Hint: Put $x = a/b$ in a suitable case of E 18.

E 33 ●● ↑ E 01 ↑ E 25 ↑ E 31

For $\alpha \le 0 \le \beta$ and for positive numbers x_i

$$\left(\frac{x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha}{n}\right)^{\frac{1}{\alpha}} \le \sqrt[n]{x_1 \ldots x_n} \le \left(\frac{x_1^\beta + x_2^\beta + \cdots + x_n^\beta}{n}\right)^{\frac{1}{\beta}}.$$

Find a proof.

Hint: Apply the inequality of E 25 for concave function $\exp(\alpha x)$ and for convex function $\exp(\beta x)$.

E 34 ●● ↑ E 25 ↑ E 26

The symbol c_α denotes the mean of order α of n positive numbers x_i, i.e.

$$c_\alpha = \left(\frac{x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha}{n}\right)^{\frac{1}{\alpha}}.$$

Prove that $\alpha \le \beta$ implies $c_\alpha \le c_\beta$.

Hint: Consider $f(t) = \log c_t$ and calculate $\frac{d}{dt} f(t)$ for $t \ne 0$.

E 35 ●● ↑ E 25 ↑ E 26

If c_α is the mean of order α of n positive numbers x_i, then

$$\lim_{\alpha \to 0} c_\alpha = \sqrt[n]{x_1 x_2 \ldots x_n}$$

and

$$\lim_{\alpha \to \infty} c_\alpha = \max(x_1 x_2 \ldots x_n).$$

5

E 36 •

Prove that

(i) $4x^2 - xy + y^2 > 0$ for all real nonzero values x, y;
(ii) $5x^2 - 4xy + 5y^2 - 12xz - 2yz + 10z^2 > 0$ for all real nonzero values x, y, z.

The left-hand side expressions here are called quadratic forms in two and three variables. Quadratic forms satisfying conditions of positivity for all real values of their variables are called positive definite forms. This concept plays an important role in linear algebra and other branches of mathematics.

E 37 •• ↑ **E 01** ↑ **E 05**

Let the polynomial $f(z) = \sum_{k=0}^{n} a_k z^{n-k}$ have complex coefficients. Then any of its zeros z satisfies the inequality $|z| \le 1 + \max|\frac{a_k}{a_0}|$. Can you prove this estimate?

Hint: If the zero z satisfies $|z| \le 1$ there is nothing to prove. For $|z| > 1$ and $\mu = \max|\frac{a_i}{a_0}|$ use the inequalities

$$\sum_{k=1}^{n} |z^{-k}| \le \sum_{k=1}^{\infty} |z^{-k}| = \frac{1}{|z| - 1}$$

and

$$|f(z)| = |a_0/z|^n \left| 1 + \sum_{k=1}^{n} (a_k/a_0) z^{-k} \right| \ge \left| \frac{a_0}{z} \right|^n \left(1 - \mu \sum_{k=1}^{n} |z^{-k}| \right).$$

E 38 • ↓ **E 40** ↓ **E 41**

The relation $\prec_F$ in the set of natural numbers is defined as follows: for $n, m \in N$ there is $n \prec_F m$ iff n divides m.
Is this relation a (partial or linear) order?

Hint: It may help to represent graphically the partial order of a given set.

E 39 ●●

Let L be the set of all points with integer coordinates in the plane. Define the binary relation $\prec_L$ as follows: $(a, b) \prec_L (x, y)$ iff $(a + b < x + y)$ or $(a + b = x + y$ and $a - b \leq x - y)$. Is this binary relation a linear order?
Consider this binary relation for the set $M \subset L$ consisting of points with nonnegative integer coordinates. Find the minimal element of M with respect to $\prec_L$. Mark each point (m, n) of M by the integer $f(m, n)$ determining its position in the considered ordering. Can you give an explicit expression for $f(m, n)$?

E 40 ●

Let X be the collection of all finite sequences of 0's and 1's. We define for $x, y \in X : x \prec_E y$ iff $y = x$ or y extends x. Is $\prec_E$ a partial order?

E 41 ●●

Let S be the set of all integer-valued sequences. We define for $a, b \in S$ two relations

$a \prec_P b$ iff $a_n \leq b_n$ for all n

and

$a \prec_A b$ iff the set $\{n : a_n > b_n\}$ is finite.

Show that both $\prec_P, \prec_A$ are partial orders. Let $e_n = (n, n, \ldots)$.

1. Does there exist an element $c \in S$ such that for all n we have $e_n \prec_P c$?
2. Does there exist an element $c \in S$ such that for all n we have $e_n \prec_A c$?

5

Supplementary Material

Definitions

A function f defined on the interval $[a, b]$ is called convex on this interval if the inequality

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y)$$

holds whenever $x, y \in [a, b]$ and $0 \leq \lambda \leq 1$.
The function f is called concave on this interval if

$$f((1-\lambda)x + \lambda y) \geq (1-\lambda) f(x) + \lambda f(y)$$

holds whenever $x, y \in [a, b]$ and $0 \leq \lambda \leq 1$.

For n positive numbers a_i the value

$$c_\alpha = \left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n} \right)^{\frac{1}{\alpha}}$$

is called their mean value of order α.
In particular, c_1 is called their arithmetic mean, c_{-1} their harmonic mean.

The expression $\sum_{i,k=1}^n a_{ik} x_i x_k$ is called a quadratic form in n variables $x_1, x_2, \ldots, x_n$. The real numbers $a_{ik} = a_{ki}$ are its coefficients and the matrix $A = [a_{ik}]$ is called its matrix.

The set A is called partially ordered if a binary relation $\prec$, called partial order, is defined satisfying the following conditions:

1. (Reflexivity): $a \prec a$;
2. (Antisymmetry): if $a \prec b$ and $b \prec a$, then $a = b$;
3. (Transitivity): if $a \prec b$ and $b \prec c$, then $a \prec c$.

If such a relation is defined for all pairs of A then the set A is called linearly ordered.

Theorems

A

If the function f is twice differentiable on the interval I, then f is convex on I iff the second derivative of f is nonnegative on I.

Plans of Solution

PE 29

1. The quantities x_i must be nonnegative.
2. For each b_k the inequality $a_{1k}x_1 + a_{2k}x_2 \le b_k$, $k = 1, 2, 3, 4$, must be respected.
3. Among all solutions of the inequalities $x_1 \ge 0$, $x_2 \ge 0$, $2x_1 + 3x_2 \le 19$, $2x_1 + x_2 \le 13$, $3x_2 \le 15$, $3x_1 \le 18$ such values of x_1, x_2 are to be found which give the function $F = 7x_1 + 5x_2$ its maximal value.
4. The graphs of the involved straight lines enable to find the solution in this case.

PE 30

1. The following equalities hold:

$$x_{11} + x_{21} = b_1, \qquad x_{12} + x_{22} = b_2, \qquad x_{13} + x_{23} = b_3,$$
$$x_{11} + x_{12} + x_{13} = a_1, \qquad x_{21} + x_{22} + x_{23} = a_2.$$

2. There are six unknowns and the rank of the system is 4. Hence two of the unknowns can be chosen arbitrarily.
3. Rewrite the equations and the function F in terms of these two, say x_{11}, x_{12}, and take into account that all the unknowns must be positive.
4. Obtain a system of six linear inequalities. Choosing x_{11}, x_{12} as the coordinate axes and F as the equation of a plane we obtain the result.

Further References

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